CONVOLUTIONS OF FIBONACCI-TYPE POLYNOMIALS OF ORDER K AND THE NEGATIVE BINOMIAL DISTRIBUTIONS OF THE SAME ORDER

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1. Introduction and Summary

Unless otherwise explicitly stated, in this paper k is a fixed positive integer, n_i $(1 \le i \le k)$ and n are nonnegative integers as specified, p and x are real numbers in the intervals $(0, 1)$ and $(0, \infty)$, respectively, $q = 1 - p$, and $[x]$ denotes the greatest integer in x. Let

$$\{F^{(k)}(x)\}_{n=0}^{\infty}$$

be the sequence of Fibonacci-type polynomials of order k, i.e.,

$$F_0^{(k)}(x) = 0, \; F_1^{(k)}(x) = 1,$$

and

$$F^{(k)}(x) = \begin{cases} x\sum_{i=1}^{n} F_{n-i}^{(k)}(x) & \text{if } 2 \le n \le k+1, \\ x\sum_{i=1}^{n} F_{n-i}^{(k)}(x) & \text{if } n \ge k+2. \end{cases} \tag{1.1}$$

This definition is due to Philippou, Georghiou, and Philippou [11] (see also [8]), who obtained the following results:

$$\sum_{n=0}^{\infty} s^n F_{n+1}^{(k)}(x) = \frac{1}{1 - xs - \cdots - xs^k} = \frac{1 - s}{1 - (1 + x)s + xs^{k+1}}, \tag{1.2}$$

$$|s| < 1/(1 + x),$$

$$F_{n+1}^{(k)}(x) = \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} x^{n_1 + \cdots + n_k}, \; n \ge 0, \tag{1.3}$$

and

$$F_{n+1}^{(k)}(x) = \sum_{i=0}^{1} (-1)^i (1 + x)^{n-i} \sum_{j=0}^{[(n-i)/(k+1)]} (-1)^j \binom{n - i - kj}{j} \times x^j (1 + x)^{-(k+1)j}, \; n \ge 0. \tag{1.4}$$

Now let N_k be a random variable which denotes the number of Bernoulli trials until the occurrence of the k^{th} consecutive success. Then

$$P(N_k = n) = p^n F_{n+1-k}^{(k)}(q/p), \; n \ge k, \tag{1.5}$$

$$P(N_k = n + k) = \sum_{i=0}^{1} (-1)^i p^{k+i} \sum_{j=0}^{[(n-i)/(k+1)]} (-1)^j \binom{n - i - kj}{j} (qp^k)^j, \; n \ge 0, \tag{1.6}$$

and

$$P(N_k = n) = \begin{cases} p^k, & n = k, \\ qp^k, & k + 1 \le n \le 2k, \\ P(N_k = n - 1) - qp^k P(N_k = n - 1 - k), & n \ge 2k + 1. \end{cases} \tag{1.7}$$

The three results above are due, respectively, to Philippou, Georghiou, and Philippou [11], Uppuluri and Patil [12], and Philippou and Makri [8]. We note, however, that expression (1.6) was implicit in the work of [10], and variants of (1.7) have also been established in [1] and [6] by different methods.

In the present paper we generalize relations (1.6) and (1.7) to two types of negative binomial distributions of order k (see Propositions 3.3 and 3.4, and Theorems 3.1 and 3.2), and we illustrate the computational usefulness of Proposition 3.3. The first type of negative binomial distribution of order k was introduced and studied in [9] and [5], while the second type was considered in [4]. Although the latter was recognized as a negative binomial distribution of order k, different from the first, it was named in [4] "compound Poisson distribution of order k" as arising from the Poisson distribution of order k by compounding. The above-mentioned propositions and theorems are stated and proved in Section 3. Their proofs depend on generalizations of expressions (1.1)-(1.5) to the $(r - 1)$-fold convolution of $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ with itself, which we proceed to discuss first. Here, and in the sequel, r is a positive integer, unless otherwise explicitly stated.

2. Convolutions of Fibonacci-Type Polynomials of Order k

Let $\{F_{n,r}^{(k)}(x)\}_{n=0}^{\infty}$ be the $(r-1)$-fold convolution of the sequence $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ with itself, i.e., $F_{0,r}^{(k)}(x) = 0$, and for $n \ge 1$,

$$F_{n,r}^{(k)}(x) = \begin{cases} F_n^{(k)}(x) & \text{if } r = 1, \\ \displaystyle\sum_{j=1}^{n} F_{j,r-1}^{(k)}(x) F_{n+1-j}^{(k)}(x) & \text{if } r \ge 2. \end{cases} \tag{2.1}$$

As a consequence of (2.1), and in view of (1.2), we get

$$\sum_{n=0}^{\infty} s^n F_{n+1,r}^{(k)}(x) = \left[\sum_{n=0}^{\infty} s^n F_{n+1}^{(k)}(x)\right]^r = \frac{1}{(1 - xs - \cdots - xs^k)^r} \tag{2.2}$$

$$= \frac{(1 - s)^r}{[1 - (1 + x)s + xs^{k+1}]^r}, \quad |s| < 1/(1 + x).$$

Expanding (2.2) as a Taylor series about $s = 0$, and following procedures similar to those of [9]-[11], we readily find the following closed formulas for $\{F_{n,r}^{(k)}(x)\}_{n=0}^{\infty}$, in terms of the multinomial and binomial coefficients, respectively.

Theorem 2.1: Let $\{F_{n,r}^{(k)}(x)\}_{n=0}^{\infty}$ be the $(r - 1)$-fold convolution of the sequence $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ with itself. Then

(a) $$F_{n+1,r}^{(k)}(x) = \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k + r - 1}{n_1, \ldots, n_k, r - 1} x^{n_1 + \cdots + n_k}, \quad n \ge 0;$$

and

(b) $$F_{n+1,r}^{(k)}(x) = \sum_{i=0}^{r}(-1)^i\binom{r}{i}(1+x)^{n-i}\sum_{j=0}^{[(n-i)/(k+1)]}(-1)^j\binom{n-i-kj}{j}$$
$$\times\binom{n-i-kj+r-1}{r-1}x^j(1+x)^{-(k+1)j}, \quad n \geq 0.$$

We also note that, if we multiply both sides of (2.2) by x^r and then differentiate them with respect to x, we obtain the following reduction formula with respect to r.

Proposition 2.1: Let $\{F_{n,r}^{(k)}(x)\}_{n=0}^{\infty}$ be the $(r-1)$-fold convolution of the sequence $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ with itself. Then

$$F_{n+1,r+1}^{(k)}(x) = \frac{d}{dx}[x^r F_{n+1,r}^{(k)}(x)]/rx^{r-1}, \quad n \geq 0.$$

We proceed next to show that $\{F_{n,r}^{(k)}(x)\}_{n=0}^{\infty}$ satisfies the following linear recurrence with variable coefficients.

Theorem 2.2: Let $\{F_{n,r}^{(k)}(x)\}$ be the $(r-1)$-fold convolution of the sequence $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ with itself, and set $F_{n,r}^{(k)}(x) = 0$ for $-k+1 \leq n \leq -1$. Then

$$F_{0,r}^{(k)}(x) = 0, \; F_{1,r}^{(k)}(x) = 1,$$

and

$$F_{n+1,r}^{(k)}(x) = \frac{x}{n}\sum_{j=1}^{k}[n+j(r-1)]F_{n+1-j,r}^{(k)}(x), \quad n \geq 1.$$

From the definition of $\{F_{n,r}^{(k)}(x)\}_{n=0}^{\infty}$, we have

$$F_{0,r}^{(k)}(x) = 0 \quad \text{and} \quad F_{1,r}^{(k)}(x) = 1. \tag{2.3}$$

Now, let $|s| < 1/(1+x)$. Noting that

$$(1 - xs - \cdots - xs^k)^{-r} = (1 - xs - \cdots - xs^k)^{-r-1}\left(1 - x\sum_{j=1}^{k}s^j\right), \tag{2.4}$$

and

$$s^j(1 - xs - \cdots - xs^k)^{-r-1} = \sum_{n=0}^{\infty}s^n F_{n+1-j,r+1}^{(k)}(x), \quad 1 \leq j \leq k, \tag{2.5}$$

by (2.2),

we get

$$\begin{aligned}\sum_{n=0}^{\infty}s^n F_{n+1,r}^{(k)}(x) &= (1 - xs - \cdots - xs^k)^{-r-1} - x\sum_{j=1}^{k}s^j(1 - xs - \cdots - xs^k)^{-r-1}, \\ &\qquad \text{by (2.2) and (2.4)}, \\ &= \sum_{n=0}^{\infty}s^n F_{n+1,r+1}^{(k)}(x) - x\sum_{j=1}^{k}\sum_{n=0}^{\infty}s^n F_{n+1-j,r+1}^{(k)}(x), \\ &\qquad \text{by (2.2) and (2.5)}, \\ &= \sum_{n=0}^{\infty}s^n\left[F_{n+1,r+1}^{(k)}(x) - x\sum_{j=1}^{k}F_{n+1-j,r+1}^{(k)}(x)\right].\end{aligned}$$

Therefore,

$$F_{n+1,r}^{(k)}(x) = F_{n+1,r+1}^{(k)}(x) - x\sum_{j=1}^{k}F_{n+1-j,r+1}^{(k)}(x), \quad n \geq 0. \tag{2.6}$$

Next, differentiating both sides of (2.2) with respect to s, we get

$$\begin{aligned}\sum_{n=1}^{\infty}ns^{n-1}F_{n+1,r}^{(k)}(x) &= rx\sum_{j=1}^{k}js^{j-1}(1 - xs - \cdots - xs^k)^{-r-1} \\ &= rx\sum_{j=1}^{k}j\sum_{n=0}^{\infty}s^n F_{n+2-j,r+1}^{(k)}(x), \text{ by (2.2) and (2.5)},\end{aligned}$$

$$= \sum_{n=1}^{\infty} s^{n-1} rx \sum_{j=1}^{k} jF_{n+1-j, r+1}^{(k)}(x),$$

which implies

$$nF_{n+1, r}^{(k)}(x) = rx \sum_{j=1}^{k} jF_{n+1-j, r+1}^{(k)}(x), \; n \geq 1. \tag{2.7}$$

Combining (2.6) and (2.7), we obtain

$$F_{n+1, r+1}^{(k)}(x) = \frac{x}{n} \sum_{j=1}^{k} (n + jr) F_{n+1-j, r+1}^{(k)}(x), \; n \geq 1,$$

which, along with (1.1) and (2.3), establishes the theorem.

Remark 2.1: Results analogous to Proposition 2.1 and Theorem 2.2 have been obtained by Horadam and Mahon [3] for convolutions of the sequence of Pell polynomials (of order 2) with itself.

3. Binomial Expressions and Recurrences for the Negative Binomial Distributions of Order k

In the present section, we employ Theorems 2.1 and 2.2 to derive binomial expressions and simple recurrences for the following two distributions of order k [4], [5], [9].

Definition 3.1: A random variable X is said to be distributed as negative binomial distribution of order k, type I, with parameter vector (r, p), to be denoted by $NB_{k, \mathrm{I}}(r, p)$, if

$$P(X = n) = p^n \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n - kr}} \binom{n_1 + \cdots + n_k + r - 1}{n_1, \ldots, n_k, r - 1} \left(\frac{q}{p}\right)^{n_1 + \cdots + n_k}, \; n \geq kr.$$

Definition 3.2: A random variable X is said to be distributed as negative binomial distribution of order k, type II, with parameter vector (r, p), to be denoted by $NB_{k, \mathrm{II}}(r, p)$, if

$$P(X = n) = p^r \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k + r - 1}{n_1, \ldots, n_k, r - 1} \left(\frac{q}{p}\right)^{n_1 + \cdots + n_k}, \; n \geq 0.$$

The negative binomial distribution of order k, type I, gives the probability that the first occurrence of r success runs of length k happens at trial n [5]. The negative binomial distribution of order k, type II, arises as a gamma compound Poisson distribution of order k. More precisely, if we use $CP_k(r, \alpha)$ to denote the (gamma) compound Poisson distribution of order k with parameter vector (r, α) [4], we note that

$$NB_{k, \mathrm{II}}(r, p) = CP_k(r, \alpha) \text{ for } p = \alpha/(\alpha + k).$$

The fact that $CP_k(r, \alpha)$ is a negative binomial distribution of order k, albeit different from $NB_{k, \mathrm{I}}(r, p)$, was already mentioned in [4] by Philippou, who named the new distribution, however, "compound Poisson distribution of order k" as arising from the Poisson distribution of order k by compounding.

As a consequence of Theorem 2.1(a) and Definitions 3.1 and 3.2, respectively, we have the following relationships.

Proposition 3.1: Let X be a random variable distributed as $NB_{k,\mathrm{I}}(r, p)$ and let $\{F_{n,r}^{(k)}(x)\}_{n=0}^{\infty}$ be the $(r-1)$-fold convolution of $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ with itself. Then

$$P(X = n) = p^n F_{n+1-kr,\,r}^{(k)}(q/p), \quad n \geq kr.$$

Proposition 3.2: Let X be a random variable distributed as $NB_{k,\mathrm{II}}(r, p)$ and let $\{F_{n,r}^{(k)}(x)\}_{n=0}^{\infty}$ be as above. Then

$$P(X = n) = p^r F_{n+1,\,r}^{(k)}(q/k), \quad n \geq 0.$$

Combining Theorem 2.1(b) with Propositions 3.1 and 3.2, respectively, we obtain the following binomial expressions for the negative binomial distributions of order k.

Proposition 3.3: Let X be a random variable distributed as $NB_{k,\mathrm{I}}(r, p)$. Then

$$P(X = n + kr) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} p^{kr+i} \sum_{j=0}^{[(n-i)/(k+1)]} (-1)^j \binom{n-i-kj}{j} \times \binom{n-i-kj+r-1}{r-1} (qp^k)^j, \quad n \geq 0.$$

Proposition 3.4: Let X be a random variable distributed as $NB_{k,\mathrm{II}}(r, p)$. Then

$$P(X = n) = p^r \sum_{i=0}^{r} (-1)^i \binom{r}{i} \left(\frac{k+q}{k}\right)^{n-i} \sum_{j=0}^{[(n-i)/(k+1)]} (-1)^j \binom{n-i-kj}{j} \times \binom{n-i-kj+r-1}{r-1} \left[\left(\frac{q}{k}\right)\left(\frac{k}{k+q}\right)^{k+1}\right]^j, \quad n \geq 0.$$

Remark 3.1: Another binomial expression for the probabilities $P(X = n + kr)$ ($n \geq 0$) of $NB_{k,\mathrm{I}}(r, p)$ has been obtained by Charalambides [2], who employed for this purpose the truncated exponential Bell polynomials. Our expression appears to be more applicable.

Remark 3.2: For $r = 1$, Propositions 3.3 and 3.4 provide binomial expressions for the probabilities of

$$G_{k,\mathrm{I}}(p) \equiv NB_{k,\mathrm{I}}(1, p) \quad \text{and} \quad G_{k,\mathrm{II}}(p) \equiv NB_{k,\mathrm{II}}(1, p),$$

respectively. The first one implies (1.6), the main result of Uppuluri and Patil [12], since N_k is distributed as $G_{k,\mathrm{I}}(p)$ [7], [9]. The second is noted presently for the first time.

Theorem 2.2 and Proposition 2.1 imply

Theorem 3.1: Let X be a random variable distributed as $NB_{k,\mathrm{I}}(r, p)$, and set $P_n = P(X = n)$. Then

$$P_n = \begin{cases} 0, & n \leq kr - 1, \\ p^{kr}, & n = kr, \\ \dfrac{(q/p)}{n-kr} \displaystyle\sum_{j=1}^{k} [n - kr + j(r-1)] p^j P_{n-j}, & n \geq kr + 1. \end{cases}$$

Proof: For $n \leq kr - 1$, $(X = n) = \emptyset$, which implies $P_n = P(\emptyset) = 0$. For $n = kr$, Definition 3.1 gives $P_n = p^{kr}$. For $n \geq kr + 1$, we have

$$P_n = p^n F_{n-kr+1,\,r}^{(k)}(q/p), \text{ by Proposition 3.1,}$$

$$= p^n \frac{(q/p)}{n - kr} \sum_{j=1}^{k} [n - kr + j(r - 1)] F^{(k)}_{n-kr+1-j,\, r}(q/p), \text{ by Theorem 2.2,}$$

$$= \frac{(q/p)}{n - kr} \sum_{j=1}^{k} [n - kr + j(r - 1)] p^j P_{n-j}, \text{ by Proposition 3.1.}$$

For $r = 1$, Theorem 3.1 reduces to the following corollary, which implies recurrence (1.7), since N_k is distributed as $G_{k,\mathrm{I}}(p)$ [7], [9].

Corollary 3.1: Let X be a random variable distributed as $G_{k,\mathrm{I}}(p) \equiv NB_{k,\mathrm{I}}(1, p)$, and set $P_n = P(X = n)$. Then

$$P_n = \begin{cases} p^k, & n = k, \\ qp^k, & k + 1 \le n \le 2k, \\ P_{n-1} - qp^k P_{n-1-k}, & n \ge 2k + 1. \end{cases}$$

Theorem 2.2 and Proposition 2.2 imply

Theorem 3.2: Let X be a random variable distributed as $NB_{k,\mathrm{II}}(r, p)$, and set $P_n = P(X = n)$, $n \ge -k + 1$. Then

$$P_n = \begin{cases} 0, & -k + 1 \le n \le -1, \\ p^r, & n = 0, \\ \dfrac{q}{kn} \displaystyle\sum_{j=1}^{k} [n + j(r - 1)] P_{n-j}, & n \ge 1. \end{cases}$$

Proof: For $-k + 1 \le n \le -1$, $(X = n) = \emptyset$, which implies $P_n = P(\emptyset) = 0$. For $n = 0$, Definition 3.2 gives $P_n = p^r$. For $n \ge 1$, we have

$$P_n = p^r F^{(k)}_{n+1,\, r}(q/k), \text{ by Proposition 3.2,}$$

$$= p^r \frac{(q/k)}{n} \sum_{j=1}^{k} [n + j(r - 1)] F^{(k)}_{n+1-j,\, r}(q/k), \text{ by Theorem 2.2,}$$

$$= \frac{q}{kn} \sum_{j=1}^{k} [n + j(r - 1)] P_{n-j}, \text{ by Proposition 3.2,}$$

which completes the proof of the theorem.

For $r = 1$, Theorem 3.2 reduces to the following corollary.

Corollary 3.2: Let X be a random variable distributed as $G_{k,\mathrm{II}}(p) \equiv NB_{k,\mathrm{II}}(1, p)$, and set $P_n = P(X = n)$, $n \ge -k + 1$. Then

$$P_n = \begin{cases} p, & n = 0 \\ \left(\dfrac{k + q}{k}\right)^{n-1} \dfrac{pq}{k}, & 1 \le n \le k, \\ \left(\dfrac{k + q}{k}\right) P_{n-1} - \dfrac{q}{k} P_{n-1-k}, & n \ge k + 1. \end{cases}$$

4. Computational Examples

In this section we illustrate the computational usefulness of Propositions 3.3 and 3.4. Since both propositions are of the same nature, we restrict attention to Proposition 3.3 in comparison to Definition 3.1.

Example 4.1: Assume that a random variable X is distributed as $NB_{3,\mathrm{I}}(5, p)$ and we are interested in calculating $P(X = 18)$ and $P(X = 20)$.

Proposition 3.3 gives

$$P(X = 18) = \sum_{i=0}^{5} (-1)^i \binom{5}{i}\binom{7-i}{4} p^{15+i} = 35p^{15} - 75p^{16} + 50p^{17} - 10p^{18} \tag{4.1}$$

and

$$\begin{aligned} P(X = 20) &= \sum_{i=0}^{5} (-1)^i \binom{5}{i} p^{15+i} \sum_{j=0}^{[(5-i)/4]} (-1)^j \binom{5-i-3j}{j}\binom{9-i-3j}{4}(qp^3)^j \\ &= p^{15}\left[\binom{9}{4} - 2\binom{6}{4}qp^3\right] - 5p^{16}\left[\binom{8}{4} - \binom{5}{4}qp^3\right] + 10p^{17}\binom{7}{4} \\ &\qquad - 10p^{18}\binom{6}{4} + 5p^{19}\binom{5}{4} - p^{20}\binom{4}{4} \\ &= 126p^{15} - 350p^{16} + 350p^{17} - 150p^{18} + 25p^{19} - p^{20} \\ &\qquad - 30qp^{18} + 25qp^{19}. \end{aligned} \tag{4.2}$$

Alternatively, if we use Definition 3.1, we get

$$\begin{aligned} P(X = 18) &= p^{18} \sum_{\substack{n_1, n_2, n_3 \ni \\ n_1 + 2n_2 + 3n_3 = 3}} \binom{n_1 + n_2 + n_3 + 4}{n_1,\ n_2,\ n_3,\ 4}\left(\frac{q}{p}\right)^{n_1+n_2+n_3} \\ &= p^{18}\left[\binom{3+0+0+4}{3,\ 0,\ 0,\ 4}\left(\frac{q}{p}\right)^3 + \binom{1+1+0+4}{1,\ 1,\ 0,\ 4}\left(\frac{q}{p}\right)^2 + \binom{0+0+1+4}{0,\ 0,\ 1,\ 4}\left(\frac{q}{p}\right)^1\right] \\ &= 35q^3p^{15} + 30q^2p^{16} + 5qp^{17} \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} P(X = 20) &= p^{20} \sum_{\substack{n_1, n_2, n_3 \ni \\ n_1 + 2n_2 + 3n_3 = 5}} \binom{n_1 + n_2 + n_3 + 4}{n_1,\ n_2,\ n_3,\ 4}\left(\frac{q}{p}\right)^{n_1+n_2+n_3} \\ &= p^{20}\left[\binom{5+0+0+4}{5,\ 0,\ 0,\ 4}\left(\frac{q}{p}\right)^5 + \binom{3+1+0+4}{3,\ 1,\ 0,\ 4}\left(\frac{q}{p}\right)^4 + \binom{2+0+1+4}{2,\ 0,\ 1,\ 4}\left(\frac{q}{p}\right)^3\right. \\ &\qquad \left. + \binom{1+2+0+4}{1,\ 2,\ 0,\ 4}\left(\frac{q}{p}\right)^3 + \binom{0+1+1+4}{0,\ 1,\ 1,\ 4}\left(\frac{q}{p}\right)^2\right] \\ &= 126q^5p^{15} + 280q^4p^{16} + 210q^3p^{17} + 30q^2p^{18}. \end{aligned} \tag{4.4}$$

Example 4.2: Assume that a random variable X is distributed as $NB_{20,\mathrm{I}}(3, p)$ and we are interested in calculating $P(X = 80)$ and $P(X = 100)$.

Proposition 3.3 gives

$$\begin{aligned} P(X = 80) &= \sum_{i=0}^{3} (-1)^i \binom{3}{i}\binom{22-i}{2} p^{60+i} \\ &= 231p^{60} - 630p^{61} + 570p^{62} - 171p^{63} \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &P(X = 100) \\ &= \sum_{i=0}^{3} (-1)^i \binom{3}{i} p^{60+i} \sum_{j=0}^{[(40-i)/21]} (-1)^j \binom{40-i-20j}{j}\binom{42-i-20j}{2}(qp^{20})^j \end{aligned}$$

$$= p^{60}\left[\binom{42}{2} - 20\binom{22}{2}qp^{20}\right] - 3p^{61}\left[\binom{41}{2} - 19\binom{21}{2}qp^{20}\right]$$
$$+ 3p^{62}\left[\binom{40}{2} - 18\binom{20}{2}qp^{20}\right] - p^{63}\left[\binom{39}{2} - 17\binom{19}{2}qp^{60}\right]$$
$$= 861p^{60} - 2460p^{61} + 2340p^{62} - 741p^{63} - 4620qp^{80}$$
$$+ 11970qp^{81} - 10260qp^{82} + 2907qp^{83}. \tag{4.6}$$

On the other hand, Definition 3.1 does not appear to be applicable for this task without a considerable amount of computational effort, even with the aid of the computer.

In general, when k and $n - kr$ are large, Proposition 3.3 fares much better than Definition 3.1 for calculating negative binomial probabilities of order k, type I. If all probabilities up to $P(X = m)$ are needed, for some integer m ($\geq kr$), the recurrence given in Theorem 3.1 is most appropriate for calculating them.

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