SOME NEW RESULTS ON QUASI-ORTHOGONAL NUMBERS

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1. Introduction

As far as is known to this author, the term "Quasi-Orthogonality" was first introduced by K. S. Miller in [1]:

Given two sets of numbers $A(m, n)$ and $B(m, n)$ such that $m, n, s \in Z$, and $A(m, n)$, $B(m, n) = 0$ for $n < 0$, $m < 0$, and $n < m$, they are said to be quasi-orthogonal to each other if

$$\sum_{s=m}^{n} A(s, n)B(m, s) = \delta(m, n) \tag{1}$$

where $\delta(m, n)$ is the Kronecker delta.

Equivalently, we can say that if $A(n)$ is the square, and triangular matrix of elements $A(m, n)$ of n rows, and $B(n)$ the square and triangular matrix of elements $B(m, n)$ of n rows, then

$$A(n)B(n) = I, \tag{2}$$

i.e., the two matrices are inverse of each other.

H. W. Gould has compared the different aspects of quasi-orthogonality and studied some of its properties [2].

In this paper we shall be concerned with the so-called BILINEARLY RECURRENT orthogonal numbers, i.e., numbers satisfying recurrence relations of the form:

$$A(m, n) = f_1(m, n)A(m - 1, n - 1) + f_2(m, n)A(m, n - 1); \tag{3}$$

$$B(m, n) = f_3(m, n)B(m - 1, n - 1) + f_4(m, n)B(m, n - 1). \tag{4}$$

The problem to solve is the following: knowing f_1 and f_2, find f_3 and f_4, or, since the problem is symmetric, knowing f_3 and f_4, find f_1 and f_2.

So far, only the following cases have been studied:

Case 1: $f_1 = N(n)$, $f_2 = M(n)$,
$f_3 = 1/[N(m + 1)]$, $f_4 = -M(m + 1)/[N(m + 1)]$. Cf. [3].

Case 2: $f_1 = P(m)$, $f_2 = K(n) + M(m + 1)$,
$f_3 = 1/P(n)$, $f_4 = -[K(m + 1) + M(n)]/P(n)$. Cf. [3].

Other cases of quasi-orthogonal numbers have been studied but they are not of the bilinearly recurrent kind.

The final aim is to obtain a general case where the functions f_i are all of the form $f_i(m, n)$. This result has thus far been impossible to reach.

In this paper we study

Case 3: $f_1(m, n) = \alpha(m)\beta(n)$, $f_2(m, n) = \eta(n)$,
$f_3(m, n) = 1/\alpha(n)\beta(m)$, $f_4(m, n) = -\eta(m + 1)/\alpha(n)\beta(m + 1)$.

2. P-Polynomials and A-Numbers

Let J be the set of positive numbers and zero, i.e., $J = [0, Z^+]$. We assume that $m, n, k, s \in J$, and that $a(m, n)$, $b(m)$, and $c(m)$ are defined, and not equal to zero, also that $x > 0$.

Consider the polynomial

$$P(n, x) = \sum_{m=0}^{n} a(m, n)A(m, n)x^m = \prod_{k=1}^{n} [b(k) + c(k)x], \tag{5}$$

so that

$$\begin{aligned} P(n+1, x) &= \sum_{m=0}^{n+1} a(m, n+1)A(m, n+1)x^m \\ &= \prod_{k=1}^{n+1} [b(k) + c(k)x] = [b(n+1) + c(n+1)x]P(n, x) \\ &= [b(n+1) + c(n+1)x] \sum_{m=0}^{n} a(m, n)A(m, n)x^n. \end{aligned} \tag{6}$$

By comparing the coefficients of x^{m+1}, we obtain

$$\begin{aligned} a(m+1, n+1)A(m+1, n+1) &= c(n+1)a(m, n)A(m, n) \\ &\quad + b(n+1)a(m+1, n)A(m+1, n) \end{aligned}$$

or, since $a(m+1, n+1) \neq 0$,

$$\begin{aligned} A(m+1, n+1) &= c(n+1)\frac{a(m, n)}{a(m+1, n+1)} A(m, n) \\ &\quad + b(n+1)\frac{a(m+1, n)}{a(m+1, n+1)} A(m+1, n), \end{aligned}$$

or again,

$$\begin{aligned} A(m, n) &= c(n)\frac{a(m-1, n-1)}{a(m, n)} A(m-1, n-1) \\ &\quad + b(n)\frac{a(m, n-1)}{a(m, n)} A(m, n-1). \end{aligned} \tag{7}$$

This is the recurrence relation for the numbers $A(m, n)$.

3. B-Numbers

We express x^n in terms of P-polynomials as defined in Section 2, thus

$$\begin{aligned} x^n &= \sum_{s=0}^{n} \lambda(s, n)B(s, n)P(s, x) \\ &= \sum_{s=0}^{n} \lambda(s, n)B(s, n) \left[\sum_{m=0}^{n} a(m, s)A(m, s)x^m \right], \end{aligned} \tag{8}$$

where the numbers $\lambda(s, n)$ are defined, and different from zero, for $s, n \in J$, and $B(s, n)$ satisfy the conditions of Section 1.

It follows that

$$\begin{aligned} x^n &= \sum_{s=0}^{n} \sum_{m=0}^{s} \lambda(s, n)a(m, s)B(s, n)A(m, s)x^m \\ &= \sum_{m=0}^{n} x^m \left[\sum_{s=m}^{n} \lambda(s, n)a(m, s)B(s, n)A(m, s) \right], \end{aligned} \tag{9}$$

which shows that the quantity in brackets, i.e., the coefficient of x^m must be equal to δ_m^n.

To assure the quasi-orthogonality of the numbers $A(m, s)$ and $B(s, n)$ it is necessary to assume that

$$\lambda(s, n)a(m, s) = 1. \tag{10}$$

This result can be obtained in the following way:

For $m = n$, we take $\lambda(s, n)a(n, s) = 1$, i.e., $\lambda(s, n) = 1/a(n, s)$.

For $m \neq n$, i.e., for $m < n$, it is necessary to write

$$a(m, s) = a_1(m)a_2(s), \quad \lambda(s, n) = \lambda_1(s)\lambda_2(n),$$

with $\lambda_1(s) = 1/a_2(s)$, so that

$$\lambda(s, n)a(m, s) = \lambda_2(n)a_1(m),$$

which, substituted into (9), gives

$$\begin{aligned} x^n &= \sum_{m=0}^{n} \lambda_2(n)a_1(m)x^m \left[\sum_{s=m}^{n} B(s, n)A(m, s)\right] \qquad (9a) \\ &= \sum_{m=0}^{n} \lambda_2(n)a_1(m)x^m\delta_m^n, \end{aligned}$$

which is satisfied if $\lambda_2(n) = 1/a_1(n)$.

We summarize this result by writing

$$\lambda(s, n) = [1/a_2(s)]\lambda_2(n),$$

or

$$\lambda(s, n) = 1/a(n, s) = 1/a_1(n)a_2(s).$$

Under these conditions, clearly (9) can be written as

$$x^n = \sum_{m=0}^{n} x^m\delta_n^m \tag{11}$$

and

$$\sum_{s=m}^{n} B(s, n)A(m, s) = \delta_n^m. \tag{12}$$

On the other hand,

$$x^{n+1} = x^n \cdot x = \left[\sum_{s=0}^{n} \lambda(s, n)B(s, n)P(s, x)\right]x. \tag{12a}$$

Since, according to (6),

$$P(s + 1, x) = [b(s + 1 + c(s + 1)x]P(s, n), \tag{13}$$

it follows that

$$xP(s, x) = [P(s + 1, x) - b(s + 1)P(s, x)]/c(s + 1) \tag{14}$$

so that, substituting into (12a), we obtain

$$\begin{aligned} x^{n+1} &= \sum_{s=0}^{n} \lambda(s, n)B(s, n)\left[\frac{P(s + 1, x)}{c(s + 1)} - \frac{b(s + 1)}{c(s + 1)} P(s, x)\right] \qquad (15) \\ &= \sum_{s=0}^{n+1} \lambda(s, n + 1)B(s, n + 1)P(s, x). \end{aligned}$$

Comparing the coefficients of $P(s + 1, x)$, we see that

$$\lambda(s + 1, n + 1)B(s + 1, n + 1) = \frac{\lambda(s, n)}{c(s + 1)} B(s, n) - \frac{\lambda(s + 1, n)b(s + 2)}{c(s + 2)} B(s + 1, n) \tag{16}$$

or

$$B(s + 1, n + 1) = \frac{\lambda(s, n)}{\lambda(s + 1, n + 1)c(s + 1)} B(s, n) - \frac{\lambda(s + 1, n)b(s + 2)}{\lambda(s + 1, n + 1)c(s + 2)} B(s + 1, n), \tag{17}$$

or again,

$$B(s, n) = \frac{\lambda(s - 1, n - 1)}{\lambda(s, n)c(s)} B(s - 1, n - 1) - \frac{\lambda(s, n - 1)b(s + 1)}{\lambda(s, n)c(s + 1)} B(s, n - 1). \tag{18}$$

Equation (18) is a first form of the recurrence relation for the B-numbers.

4. Evaluation of $a(m, n)$

According to (4) and (7), we can write:

$$c(n)\frac{a(m - 1, n - 1)}{a(m, n)} = f_1(m, n); \tag{19}$$

$$b(n)\frac{a(m, n - 1)}{a(m, n)} = f_2(m, n). \tag{20}$$

From (20), we deduce

$$\begin{aligned} b(n)a(m, n - 1) &= f_2(m, n)a(m, n) \\ b(n - 1)a(m, n - 2) &= f_2(m, n - 1)a(m, n - 1) \\ b(n - 2)a(m, n - 3) &= f_2(m, n - 2)a(m, n - 2) \\ &\vdots \\ b(2)a(m, 1) &= f_2(m, 2)a(m, 2) \end{aligned}$$

and multiplying through and simplifying,

$$\left[\prod_{k=2}^{n} b(k)\right] a(m, 1) = a(m, n)\left[\prod_{k=2}^{n} f_2(m, k)\right]$$

or

$$a(m, n) = a(m, 1)\left[\prod_{k=2}^{n} \frac{b(k)}{f_2(m, k)}\right] \tag{21}$$

and

$$a(m - 1, n - 1) = a(m - 1, 1)\left[\prod_{k=2}^{n-1} b(k)/f_2(m - 1, k)\right]. \tag{22}$$

Substituting (21) and (22) into (19), we obtain

$$c(n)a(m - 1, 1)\left[\prod_{k=2}^{n-1} b(k)/f_2(m - 1, k)\right]$$

$$= a(m, 1)\left[\prod_{k=2}^{n} b(k)/f_2(m, k)\right] f_1(m, n)$$

which, after simplification, gives

$$a(m, 1) = a(m-1, 1)[c(n)/b(n)]$$
$$\cdot \left[\prod_{k=2}^{n-1} f_2(m, k)/f_2(m-1, k)\right][f_2(m, n)/f_1(m, n)] \tag{23}$$

or

$$a(m, 1) = a(m-1, 1)\Omega(m), \tag{24}$$

since the left-hand member of (23) is independent of n, i.e.,

$$\Omega(m) = [c(n)/b(n)]\left[\prod_{k=2}^{n-1} f_2(m, k)/f_2(m-1, k)\right][f_2(m, n)/f_1(m, n)]. \tag{25}$$

To eliminate n in the right-hand member of (25), we assume that

$$f_1(m, n) = \alpha(m)\beta(n), \text{ and } f_2(m, n) = \delta(m)\eta(n).$$

Equation (25) can then be written as

$$\Omega(m) = [c(n)/b(n)][\delta(m)/\delta(m-1)]^{n-2}[\delta(m)\eta(n)/\alpha(m)\beta(n)].$$

In order to have the right-hand side independent of n, it is necessary to assume that

$$[c(n)/b(n)][\eta(n)/\beta(n)] = A = \text{Const.}, \tag{26}$$

and

$$\delta(m)/\delta(m-1) = 1, \tag{27}$$

i.e., $\delta(m) = B = \text{Const.}$ We may also assume that $A = B = 1$, i.e.,

$$f_2(m, n) = f_2(n) = \eta(n), \tag{28}$$

$$[c(n)/b(n)][\eta(n)/\beta(n)] = 1. \tag{29}$$

It follows that $\Omega(m) = 1/\alpha(m)$ and, returning to (24), we can write

$$\begin{aligned} a(m, 1) &= a(m-1)/\alpha(m) \\ a(m-1, 1) &= a(m-2)/\alpha(m-1) \\ a(m-2, 1) &= a(m-3)/\alpha(m-2) \\ &\vdots \\ a(2, 1) &= a(1, 1)/\alpha(2), \end{aligned}$$

and multiplying through, we obtain

$$a(m, 1) = a(1, 1)\left[\prod_{j=2}^{m} 1/\alpha(j)\right]. \tag{30}$$

Substituting (30) into (21), we obtain

$$a(m, n) = a(m, 1)\prod_{k=2}^{n} b(k)/f_2(m, k) = a(1, 1)\prod_{j=2}^{m}\frac{1}{\alpha(j)}\prod_{k=2}^{n}\frac{b(k)}{\eta(k)}. \tag{31}$$

In the following examples we shall show how the results so obtained can be used to solve the proposed problem.

5. Example I

Given $A(m + 1, n + 1) = mnA(m, n) + A(m + 1, n)$, which we rewrite in the form of (4),

$$A(m, n) = (m - 1)(n - 1)A(n - 1, n - 1) + A(m, n - 1),$$

so that $f_1 = (m - 1)(n - 1)$, i.e., $\alpha(m) = m - 1$, $\beta(n) = n - 1$, $f_2 = \eta(n) = 1$.
Equation (26) gives

$$c(n)/b(n) = \beta(n)/\eta(n) = n - 1,$$

and from (31) we obtain, with $a(1, 1) = 1$,

$$a(m, n) = \prod_{j=2}^{m} \frac{1}{j - 1} \prod_{k=2}^{n} b(k) = X(n)/(m - 1)!, \quad X(n) = \prod_{k=2}^{n} b(k).$$

From (10), it follows that, since $\lambda(s, n)a(m, s) = 1$,

$$\lambda(s, n) = (n - 1)!/X(s).$$

From (18), we obtain

$$\begin{aligned} f_3 &= \lambda(s - 1, n - 1)/\lambda(s, n)c(s) \\ &= [(n - 2)!/X(s - 1)][X(s)/(n - 1)!c(s)]. \end{aligned}$$

As we have shown in this example, $c(n)/b(n) = n - 1$, so $c(n) = (n - 1)b(n)$ and $f_3 = 1/(n - 1)(s - 1)$. Again, from (18), we obtain $f_4 = -1/s(n - 1)$. It follows that the B-numbers satisfy the relation

$$B(s, n) = [1/(n - 1)(s - 1)]B(s - 1, n - 1) - [1/(n - 1)s]B(s, n - 1).$$

For $A(1, 1) = B(1, 1) = 1$, we present a table of the A- and B-numbers:

	$A(m, n)$					$B(m, n)$				
$n \backslash m$	1	2	3	4	5	1	2	3	4	5
1	1					1				
2	1	1				-1	1			
3	1	3	4			$\frac{1}{2}$	$-\frac{3}{4}$	$\frac{1}{4}$		
4	1	6	22	36		$-\frac{1}{6}$	$\frac{7}{24}$	$-\frac{11}{72}$	$\frac{1}{36}$	
5	1	10	70	300	576	$\frac{1}{24}$	$-\frac{5}{64}$	$\frac{85}{1728}$	$\frac{-25}{1728}$	$\frac{1}{576}$

6. Evaluation of f_3 and f_4

As we have seen in Section 4, it is necessary to assume that

$$f_1(m, n) = \alpha(m)\beta(n) \quad \text{and} \quad f_2(m, n) = \eta(n).$$

From (31), $a(m, n)$, and (10) and its consequences, it follows that $\lambda(s, n) = 1/a(n, s)$. Thus

$$\lambda(s, n) = \left[\prod_{j=2}^{n} \alpha(j)\right]\left[\prod_{k=2}^{n} \eta(k)/b(k)\right]. \tag{32}$$

Then it follows from (18) that

$$f_3(s, n) = \lambda(s - 1, n - 1)/\lambda(s, n)c(s) = 1/\alpha(n)\beta(s) \tag{33}$$

and

$$f_4(s, n) = -\lambda(s, n - 1)b(s + 1)/\lambda(s, n)c(s + 1)$$
$$= -\eta(s + 1)/\alpha(n)\beta(s + 1). \tag{34}$$

The results of Example I can be checked easily using (33) and (34).

7. Example II

Given

$$A(m + 1, n + 1) = \frac{n^2}{m} A(m, n) + A(m + 1, n).$$

We rewrite this in the form of (3), i.e.,

$$A(m, n) = [(n - 1)^2/(m - 1)]A(m - 1, n - 1) + A(m, n - 1).$$

It follows that

$$f_1(m, n) = \alpha(m)\beta(n) = (n - 1)^2/(m - 1),$$
$$f_2 = 1,$$
$$f_3(m, n) = (n - 1)/(m - 1)^2,$$

and $f_4(m, n) = -(n - 1)/m^2$,

so that

$$B(m, n) = [(n - 1)/(m - 1)^2]B(m - 1, n - 1) - [(n - 1)/m^2]B(m, n - 1).$$

For $A(1, 1) = 1$, we give here the values of the A- and B-numbers for $m, n \le 5$.

	$A(m, n)$					$B(m, n)$				
$n \backslash m$	1	2	3	4	5	1	2	3	4	5
1	1					1				
2	1	1				-1	1			
3	1	5	2			2	$-\frac{5}{2}$	$\frac{1}{2}$		
4	1	14	$\frac{49}{2}$	6		-6	$\frac{63}{8}$	$-\frac{49}{24}$	$\frac{1}{6}$	
5	1	30	$\frac{273}{2}$	$\frac{410}{3}$	24	24	$-\frac{255}{8}$	$\frac{1897}{216}$	$-\frac{205}{216}$	$\frac{1}{24}$

References

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