rational numbers using their common denominator $u_{r,n}$, the numerators form the sequence $\{u_{r,n+i}\}_{i=-r}^{r}$.

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NOTE ON A FAMILY OF FIBONACCI-LIKE SEQUENCES

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In [2] P. Asveld gave a solution to the recurrence relation

$$G_n = G_{n-1} + G_{n-2} + \sum_{j=0}^{k} \alpha_j n^j \text{ with } G_0 = G_1 = 1. \tag{1}$$

In [2] we showed that the solution to the recurrence relation

$$G_n = G_{n-1} + G_{n-2} + S_n,\ G_1 = S_1,\ G_2 = S_1 + S_2, \tag{2}$$

where S_n is the n^{th} term of *any* sequence $\{S_n\} \equiv S$, is given by the n^{th} term of the convolution of the Fibonacci sequence F with the sequence S. That is, the solution of (2) can be expressed as

$$G_n = (F * S)_n,$$

using $*$ to mean *convolution*.

This note shows how Asveld's family can be dealt with by the convolution technique, using generating functions. Although we do not work through the details in the note, it is clear that a comparison of the two final solutions would yield interesting identities relating Asveld's tabulated polynomials and coefficients, and the coefficients from our solution.

Solution Method

Comparing (1) and (2), we see that the sequence on the right-hand side is:

$$S \equiv \{S_n\} = \sum_{j=0}^{k} \alpha_j n^j;$$

but the initial conditions differ since G_0, G_1 both equal 1 rather than S_1 and $(S_1 + S_2)$, respectively. However, it may quickly be ascertained that with $G_0 = G_1 = 1$ Asveld's equation is satisfied by

$$G_n = F_{n+1} + (F * S')_{n-1}, \text{ where } S' = \{s_2, s_3, s_4, \ldots\}. \tag{3}$$

Now the generating function of F is $f(x) = 1/(1 - x - x^2)$. To find the generating function of S', we note that α_0 is generated by $\alpha_0/(1 - x)$, and $\alpha_j n^j$ by

$$\alpha_j \frac{d}{dx}(xg_{j-1}(x)) \text{ for } j = 1, \ldots, k,$$

where $g_i(x)$ refers to the generating function of n^i and $g_0(x) = 1/(1 - x)$. It follows that the generating function of S' is:

$$g(x) = \frac{1}{x}\left[\frac{\alpha_0}{1 - x} + \frac{\alpha_1}{(1 - x)^2} + \frac{\alpha_2(1 + x)}{(1 - x)^3} + \cdots + \alpha_k \frac{d}{dx}\Big(xg_{k-1}(x)\Big) - \sum_{j=0}^{k} \alpha_j\right]. \tag{4}$$

Finally, from (3) and (4), we know that the solution of (1) is, for $n \geq 2$:

$$G_n = F_{n+1} + C_{n-2}, \tag{5}$$

where C_{n-2} is the coefficient of x^{n-2} in the product of generating functions $f(x)$, $g(x)$, with $G_0 = G_1 = 1$.

Comparison of Solutions

As stated above, we do not wish in this note to go into the algebraic detail necessary to make a full comparison of the two types of solution. It will be instructive, however, to show the two solutions with a small value of k. We shall set $k = 2$, and then $S_n = \alpha_0 + \alpha_1 n + \alpha_2 n^2$. The solutions are:

Asveld's Solution:

$$G_n = (1 + a_{00}\alpha_0 + a_{01}\alpha_1 + a_{02}\alpha_2)F_{n+1} + \lambda_2 F_n - \sum_{j=0}^{2} \alpha_j p_j(n), \tag{6}$$

where

$$p_j(n) = \sum_{i=0}^{j} a_{ij} n^i \quad \text{and} \quad \lambda_2 = \alpha_1 + (1 + a_{12})\alpha_2,$$

and the coefficients a_{ij}

$$\left.\begin{aligned} a_{ii} &= 1 \\ a_{ij} &= -\sum_{m=i+1}^{j} \beta_{im} a_{mj}, \text{ if } j > i \end{aligned}\right\} \text{with } \beta_{im} = \binom{m}{i}(-1)^{m-i}(1 + 2^{m-i}).$$

Asveld [1] tabulates the coefficients of the α_j's in (6), and with these coefficients equation (6) reduces to the following:

$$G_n = (1 + \alpha_0 + 3\alpha_1 + 13\alpha_2)F_{n+1} + (\alpha_1 + 7\alpha_2)F_n - [\alpha_0 + (n + 3)\alpha_1 + (n^2 + 6n + 13)\alpha_2], \tag{7}$$

where $\{F_n\}$ is the Fibonacci sequence $\{1, 1, 2, 3, 5, \ldots\}$.

Turner's Solution:

For $n \geq 2$, from (5) we see that

$$G_n = F_{n+1} + C_{n-2},$$

where C_{n-2} is the coefficient of x^{n-2} in the expansion of

$$\frac{1}{x(1 - x - x^2)}\left[\frac{\alpha_0}{1 - x} + \frac{\alpha_1}{(1 - x)^2} + \frac{\alpha_2(1 + x)}{(1 - x)^3} - (\alpha_0 + \alpha_1 + \alpha_2)\right]$$

$$= (1 - x - x^2)^{-1}(1 - x)^{-3}[(\alpha_0 + 2\alpha_1 + 4\alpha_2) - (2\alpha_0 + 3\alpha_1 + 3\alpha_2)x + (\alpha_0 + \alpha_1 + \alpha_2)x^2].$$

This gives

$$G_n = F_{n+1} + \alpha_0(a - 2b + c) + \alpha_1(2a - 3b + c) + \alpha_2(4a - 3b + c), \tag{8}$$

where $a = (F * B)_{n-1}$, $b = (F * B)_{n-2}$, and $c = (F * B)_{n-3}$, with F the Fibonacci sequence and B the sequence of binomial coefficients

$$\binom{2}{0}, \binom{3}{1}, \binom{4}{2}, \ldots, \binom{n + 1}{n - 1}, \ldots .$$

[N.B. the expressions $(F * B)_i$ are to be set to zero if $i \leq 0$.]

Corresponding coefficients in (7) and (8) may now be compared, and, as promised above, interesting identities result. Thus:

Coefficients of α_0: $F_{n+1} - 1 = a - 2b + c$;

Coefficients of α_1: $3F_{n+1} + F_n - (n + 3) = 2a - 3b + c$;

Coefficients of α_2: $13F_{n+1} + 7F_n - (n^2 + 6n + 13) = 4a - 3b + c$.

These in themselves are identities relating the Fibonacci terms and the convolutions with binomial coefficients.

Solving the three equations for a, b, and c, and taking the sum $a + b + c$, leads to the identity

$$\sum_{i=1}^{3}(F * B)_{n-i} \equiv 2F_{n+5} - \tfrac{1}{2}(3n^2 + 9n + 20). \tag{9}$$

Using (9) we can obtain

$$(F * B)_n - (F * B)_{n-3} \equiv 2F_{n+4} - 3(n + 2). \tag{10}$$

Then, setting $n = 3i - 2$ in (10) and summing over $i = 1, 2, 3, \ldots, N$, we obtain

$$(F * B)_{3N-2} \equiv F_{3N+4} - \tfrac{3}{2}(3N^2 + 3N + 2). \tag{11}$$

Similar identities may be obtained for $(F * B)_{3N-1}$ and $(F * B)_{3N}$.

Clearly, repeating these procedures for $k = 3, 4, \ldots$ would lead to more and more complex identities of this type.

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MORE ON THE FIBONACCI PSEUDOPRIMES

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1. Generalities

The idea of writing this note was triggered by the necessity that occurred in the course of our research job, of expressing the quantity $x^n + y^n$ (x and y arbitrary quantities, n a nonnegative integer) in terms of powers of xy and $x + y$. Such expressions, commonly referred to as *Waring formulae*, are given in high school books and others (e.g., see [1]) only for the first few values of n, namely

$$\begin{cases} x^0 + y^0 = 2 \\ x^1 + y^1 = x + y \\ x^2 + y^2 = (x + y)^2 - 2xy \\ x^3 + y^3 = (x + y)^3 - 3xy(x + y) \\ x^4 + y^4 = (x + y)^4 - 4xy(x + y)^2 + 2(xy)^2. \end{cases} \tag{1.1}$$

Without claiming the novelty of the result, we found (see [2]) the following general expression

$$x^n + y^n = \sum_{k=0}^{[n/2]} (-1)^k C_{n,k} (xy)^k (x + y)^{n-2k}, \tag{1.2}$$

where

$$\begin{cases} C_{0,0} = 2 \\ C_{n,k} = \dfrac{n}{n - k}\dbinom{n - k}{k} = nB_{n,k} \quad (n \geq 1) \end{cases} \tag{1.3}$$

and $[a]$ denotes the greatest integer not exceeding a.

Several interesting combinatorial and trigonometrical identities emerge (see [2]) from certain choices of x and y in (1.2). In particular, sensing Lucas numbers L_n on the left-hand side of (1.2) is quite natural for a Fibonacci fan. In fact, replacing x and y by $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$, respectively, we get

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