TWO-SIDED GENERALIZED FIBONACCI SEQUENCES

Peter C. Fishburn
Andrew M. Odlyzko
AT&T Bell Laboratories, Murray Hill, NJ 07974

Fred S. Roberts
Rutgers University, New Brunswick, NJ 08903

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1. Introduction

This paper investigates a concept called a *two-sided generalized Fibonacci sequence* (TGF) that was motivated by problems of uniqueness in measurement representations [2-4, 6-8]. The particular context that gives rise to TGFs is finite algebraic difference measurement [2, 6-8]. For simplicity, suppose that $n + 1$ objects $a_1, \ldots, a_{n+1}$ are linearly ordered by a real-valued function u as

$$u(a_1) < u(a_2) < \cdots < u(a_{n+1})$$

and that comparisons can be made between positive differences $u(a_j) - u(a_i)$, $i < j$. In measurement theory, we are sometimes concerned with conditions which guarantee that the u values are unique up to a positive affine transformation

$$u \to \alpha u + \beta, \ \alpha > 0.$$

Let $d_i > 0$ be defined by

$$d_i = u(a_{i+1}) - u(a_i).$$

Then we search for conditions which guarantee that the d_i are unique up to multiplication by a positive constant α. Each *equality*-of-differences comparison yields an equation of the form

$$d_i + d_{i+1} + \cdots + d_j = d_k + d_{k+1} + \cdots + d_\ell, \ 1 \le i \le j < k \le \ell \le n,$$

in the variables d_i. If there are $n - 1$ linearly independent equations of this type that have a strictly positive solution, then their solution by positive d_i is unique up to multiplication of every d_i by the same positive constant. For example, the three equations

$$d_1 = d_2, \quad d_2 + d_3 = d_4, \quad d_1 + d_2 = d_3 + d_4$$

have solution $d_1^* \ldots d_4^* = 2213$, and if $d_1' \ldots d_4'$ is any other positive solution then there is a $\lambda > 0$ such that $d_i' = \lambda d_i^*$ for each i. We refer the interested readers to [2] for additional discussion of this type of uniqueness in the general algebraic difference setting.

A *TGF* is a finite sequence of positive integers constructed by starting with a 1 and adding terms one by one at either end of the sequence S constructed thus far so that each new term equals the sum of one or more contiguous terms on the *end* of S at which the new term is placed. A new term v added to $S = x_1 \ldots x_m$ produces either $vx_1 \ldots x_m$ with

$$v \in \{x_1, x_1 + x_2, \ldots, x_1 + \cdots + x_m\}$$

or $x_1 \ldots x_m v$ with

$$v \in \{x_m, x_m + x_{m-1}, \ldots, x_m + \cdots + x_1\}.$$

TGFs arise from specialized sets of equations of the type described in the preceding paragraph. One example for $n = 4$ is 2114, which is the unique positive solution (up to multiplication by a positive constant) to

$$d_2 = d_3, \quad d_1 = d_2 + d_3, \quad d_4 = d_1 + d_2 + d_3.$$

Although many unique solutions to equations for the general algebraic difference setting do not correspond to TGFs, as is true for

$$d_1^* \ldots d_4^* = 2213,$$

two-sided generalized Fibonacci sequences constitute an important subset of all such unique solutions, and it is this subset that we study here.

Let T_n denote the set of all n-term TGFs, and let $t_n = |T_n|$. Then

$$T_1 = \{1\}, \quad T_2 = \{(1, 1)\} = \{11\}, \quad T_3 = \{111, 112, 211\},$$

$$T_4 = \{1111, 1112, 1113, 1122, 1123, 1124, 2111, 2112, 2114, 2211, 3111, 3211, 4112, 4211\},$$

and so forth, with $t_1 = t_2 = 1$, $t_3 = 3$, $t_4 = 14$, and, as we shall see, $t_5 = 85$, $t_6 = 626, \ldots$. We note that every TGF for $n \geq 2$ has the *monotonicity property*, which means that there is a subsequence of two or more contiguous 1's and the sequence is nondecreasing in both directions away from that subsequence. Given any finite integer sequence

$$b_j \ldots b_2 b_1 1 \ldots 1 a_1 a_2 \ldots a_k$$

with the monotonicity property, a simple outside-in algorithm identifies whether it is a TGF. At each step of the algorithm, we ask whether a largest end term is the sum of a contiguous block of terms next to it. If not, the sequence is not a TGF; else delete that end term and repeat the question. If deletions leave only 1's, the sequence is a TGF.

We close this section by summarizing our main results. Our first main counting result is the nonlinear recurrence

$$t_{n+1} = 2nt_n - (n - 1)^2 t_{n-1} \text{ for } n \geq 2,$$

which has the Fibonacci feature that each new term in

$$(t_1, t_2, \ldots) = (1, 1, \ldots)$$

is determined from its two immediate predecessors. Since the t_n sequence is not in Sloane's book [10] and has not been brought to that author's attention by others (N. J. A. Sloane, personal communication), it may not have been studied previously.

The recurrence implies that

$$(\sqrt{n} + 1/2)^2 - 1/\sqrt{n} < \frac{t_{n+1}}{t_n} < (\sqrt{n} + 1/2)^2 \text{ for } n \geq 2.$$

This gives nice bounds on the ratio of successive t_n and indicates the growth rate of the t_n sequence. We omit the proof of these bounds, which follow without great difficulty from the recurrence by induction, algebraic manipulation, and subsidiary inequalities such as

$$1/2 < \sqrt{n}(\sqrt{n} - \sqrt{n - 1}).$$

Our other main result for t_n is an asymptotic estimate obtained from the exponential generating function

$$F(x) = \sum_{n=1}^{\infty} \frac{t_n x^{n-1}}{(n - 1)!}.$$

We prove that

$$F(x) = \frac{e^{\frac{1}{1-x}}}{1-x}\left[\frac{1}{e} - \int_{y=0}^{x} \frac{e^{-\frac{1}{1-y}}}{1-y}\,dy\right]$$

and use this to obtain

$$t_n \sim K(n-1)!e^{2\sqrt{n}}/n^{1/4},$$

where $K = K_1\sqrt{e/\pi}/2$ and

$$K_1 = \frac{1}{e} - \int_{y=0}^{1} \frac{e^{-\frac{1}{1-y}}}{1-y}\,dy = 0.148495\ldots .$$

The ratios of successive values of this approximation of t_n lie well within the bounds of the preceding paragraph. The generating function can also be used to obtain a fuller asymptotic approximation to t_n.

The results for t_n are proved in the next section. Section 3 examines $f(k_1, \ldots, k_m)$, the length of a shortest TGF that contains at least one permutation of the positive integer sequence $(k_1, \ldots, k_m)$ as a (not necessarily contiguous) subsequence. We note first that $f(k_1, \ldots, k_m)$ is always defined for $m \le 4$ but can be undefined for $m \ge 5$ because no TGF has a permutation of $k_1, \ldots, k_m$ as a subsequence. We then show for a single integer $k \ge 2$ that

$$f(k) = \lceil \log_2 k \rceil + 2,$$

where $\lceil x \rceil$ is the smallest integer at least as great as x. This result is followed by a proof that, when $k_1 \le k_2 \le k_3 \le k_4$, $f(k_1, k_2, k_3, k_4) - f(k_2, k_3, k_4)$ can be arbitrarily large. We do not know whether the same thing holds for $f(k_1, k_2, k_3) - f(k_2, k_3)$ or for $f(k_1, k_2) - f(k_2)$ when $k_1 \le k_2 \le k_3$, but conjecture that $f(k_1, k_2) \le f(k_2) + 1$.

The paper concludes with remarks on open problems and generalizations.

2. Counting TGFs

Theorem 1: $t_1 = 1$, $t_2 = 1$, and $t_{n+1} = 2nt_n - (n-1)^2 t_{n-1}$ for $n \ge 2$.

Proof: Each TGF $x_1 \ldots x_n$ in T_n yields n left extensions $vx_1 \ldots x_n$ in T_{n+1} for the n different values in

$$\{x_1,\ x_1 + x_2,\ \ldots,\ x_1 + \cdots + x_n\}.$$

It also yields n right extensions $x_1 x_2 \ldots x_n v$ in T_{n+1} for the n different values in

$$\{x_n,\ x_n + x_{n-1},\ \ldots,\ x_n + \cdots + x_1\}.$$

Thus, T_n induces nt_n distinct members of T_{n+1} by left extension and nt_n distinct members of T_{n+1} by right extension. But the $2nt_n$ total can contain duplications between left and right extensions.

Call a sequence in T_{n+1} a *sequence of duplication* if it arises from *both* a left extension and a right extension of sequences in T_n. Consider the condition

$$z_2 \ldots z_n \in T_{n-1},\ z_1 = z_2 + \cdots + z_j \text{ for some } 2 \le j \le n, \quad \text{(A)}$$
$$\text{and } z_{n+1} = z_n + \cdots + z_k \text{ for some } 2 \le k \le n.$$

If (A) holds, then $z_1 z_2 \ldots z_n z_{n+1}$ is clearly a sequence of duplication, since $z_1 \ldots z_n$ and $z_2 \ldots z_{n+1}$ are in T_n.

Conversely, if $z_1 \ldots z_{n+1}$ is a sequence of duplication, then (A) holds. To see this, suppose

$$z_1 \ldots z_{n+1} = ax_1 \ldots x_n = y_1 \ldots y_n b$$

with $x_1 \ldots x_n$ and $y_1 \ldots y_n$ in T_n,

$$a = x_1 + \cdots + x_j \text{ for some } 1 \le j \le n, \text{ and}$$

$$b = y_n + \cdots + y_k \text{ for some } 1 \le k \le n.$$

We cannot have $a = x_1 + \ldots + x_n$, since otherwise $y_1 > y_2 + \ldots + y_n$, contradicting $y_1 \ldots y_n \in T_n$. Similarly, b cannot equal $y_n + \ldots + y_1$. We can conclude that (A) holds for $z_1 = a$ and $z_{n+1} = b$, provided that we can show that $S = z_2 \ldots z_n$ is in T_{n-1}. Suppose, to the contrary, that $S \notin T_{n-1}$. Then

$$x_k = x_{k+1} + \cdots + x_n \text{ for some } k \le n - 1.$$

If this is true only for $k = n - 1$, then x_n can be the last term added in the construction of $x_1 \ldots x_n$ so that its deletion leaves member $x_1 \ldots x_{n-1} = S$ of T_{n-1}. Hence, we suppose that

$$x_k = x_{k+1} + \cdots + x_n \text{ for some } k \le n - 2.$$

By a symmetric argument for $y_1 \ldots y_n$, $S \notin T_{n-1}$ implies that

$$y_j = y_1 + \cdots + y_{j-1} \text{ for some } j \ge 3.$$

With k and j as just noted, $x_k = z_{k+1}$, $y_j = z_j$, and the monotonicity property for $z_1 \ldots z_{n+1}$ requires that there be some 1's to the left of z_j and some 1's to the right of z_{k+1}. Therefore, $k + 1 < j$. But then $z_{k+1} > z_j$ $(x_k > y_j)$, since z_{k+1} is a sum of terms that include z_j, and $z_j > z_{k+1}$ $(y_j > x_k)$, since z_j is a sum of terms that include z_{k+1}. We therefore have a contradiction and conclude that $S \in T_{n-1}$.

We have shown that (A) holds if and only if $z_1 \ldots z_{n+1}$ is a sequence of duplication. Since for every member of T_{n-1} each of z_1 and z_{n+1} can be chosen independently in $n - 1$ ways to satisfy (A), there are precisely $(n - 1)^2 t_{n-1}$ sequences of duplication. Each of these corresponds to one left extension and one right extension from T_n. Therefore,

$$t_{n+1} = 2nt_n - (n - 1)^2 t_{n-1}. \quad \square$$

A simple application of Theorem 1 shows that

$$t_5 = 85,\ t_6 = 626,\ t_7 = 5387,\ t_8 = 52{,}882,$$

$$t_9 = 582{,}149,\ t_{10} = 7{,}094{,}234,\ t_{11} = 94{,}730{,}611,\ \ldots .$$

Theorem 2: $t_n \sim (n - 1)!K_1\sqrt{e/\pi}\,e^{2\sqrt{n}}/(2n^{1/4})$, where

$$K_1 = \frac{1}{e} - \int_{y=0}^{1} \frac{e^{-\frac{1}{1-y}}}{1 - y}\,dy = 0.148495\ldots .$$

Proof: The proof is based on the saddle point method of asymptotic analysis described, for example, in de Bruijn [1]. As we note shortly, the main step in the proof is covered by results of Hayman [5].

We begin with the recurrence of Theorem 1 and form the exponential generating function

$$F(x) = \sum_{n=1}^{\infty} \frac{t_n x^{n-1}}{(n - 1)!}.$$

Using the recurrence, we get

$$F'(x)(1-x)^2 - F(x)(2-x) = -1.$$

We solve this linear differential equation by a standard method to obtain

$$F(x) = \frac{e^{\frac{1}{1-x}}}{1-x}\left[K_1 + \int_x^1 \frac{e^{-\frac{1}{1-y}}}{1-y}\,dy\right],$$

where K_1 is as defined in Theorem 2.

Ignoring $\int_x^1 \ldots dy$ for the moment, we use the saddle point method to obtain the asymptotic estimate of the coefficient c_n of x^n in the power series expansion of $e^{1/(1-x)}/(1-x)$. It follows from Hayman [5] (and by our independent verification) that

$$c_n \sim \frac{1}{2}\sqrt{e/\pi}\,e^{2\sqrt{n}}/n^{1/4}.$$

Since $c_n/c_{n-1} \to 1$ and (see below) $\int_x^1 \ldots dy$ is insignificant compared to K_1, we conclude that

$$t_n/(n-1)! \sim K_1\frac{1}{2}\sqrt{e/\pi}\,e^{2\sqrt{n}}/n^{1/4}$$

as claimed in Theorem 2.

To show that the $\int_x^1 \ldots dy$ part of $F(x)$ can be ignored asymptotically, we first extend this part of $F(x)$ to the complex plane by defining

$$g(z) = \frac{e^{\frac{1}{1-z}}}{1-z}\int_z^1 \frac{e^{-\frac{1}{1-u}}}{1-u}\,du = \frac{1}{1-z}\int_z^1 \frac{e^{\frac{z-u}{(1-z)(1-u)}}}{1-u}\,du = \sum_{n=0}^{\infty} d_n z^n.$$

By Cauchy's integral equation,

$$d_n = \frac{1}{2\pi i}\oint_{|z|=r}\frac{g(z)}{z^n}\,\frac{dz}{z},$$

and therefore,

$$|d_n| \le \frac{\max|g(z)|}{|z|^n} = \frac{\max|g(z)|}{r^n},$$

where $r = 1 - 1/\sqrt{n}$ and the max is taken on the circle $|z| = r$. We shall show that

$$|g(z)| = O(\sqrt{n}) \text{ for all } z \text{ with } |z| = r.$$

It then follows that

$$|d_n| = O(\sqrt{n}\,e^{\sqrt{n}})$$

and hence that

$$\frac{|d_n|}{K_1 c_n} = O(n^{3/4}/e^{\sqrt{n}}) \to 0.$$

Therefore, the total coefficient of x^n in the power series expansion of $F(x)$ is $\sim K_1 c_n$.

To obtain

$$|g(z)| = O(\sqrt{n}) \text{ on } |z| = r,$$

we begin with the second integral expression of $g(z)$ in the preceding paragraph and define α by

$$u = 1 - \alpha(1-z)$$

to obtain

$$g(z) = \frac{1}{1-z}\int_{\alpha=0}^{1} e^{(1-1/\alpha)/(1-z)}\frac{d\alpha}{\alpha}.$$

Since $\mathrm{Re}(1/(1-z)) = (1-\mathrm{Re}(z))/|1-z|^2$ and $1 - 1/\alpha < 0$, this yields

$$|g(z)| \leq \frac{1}{|1-z|}\int_{\alpha=0}^{1} e^{(1-1/\alpha)(1-\mathrm{Re}(z))/|1-z|^2}\frac{d\alpha}{\alpha}.$$

With $z = r(\cos\theta + i\sin\theta)$ in polar coordinates,

$$|1-z| = (1 - 2r\cos\theta + r^2)^{1/2}.$$

This is minimized at $\theta = 0$, so

$$\min|1-z| = 1 - r = 1/\sqrt{n}.$$

Therefore,

$$\max(1/|1-z|) = \sqrt{n}.$$

Moreover, $\mathrm{Re}(1/(1-z))$ is easily seen to be maximized at $\theta = \pi$, where it equals $1/(1+r)$, or about $1/2$. Let $\beta > 0$ be a constant less than $\mathrm{Re}(1/(1-z))$ for all $|z| = r$. Since $1 - 1/\alpha$ in the exponent of the preceding integral is negative, it follows that

$$|g(z)| = O\left(\sqrt{n}\int_{\alpha=0}^{1} e^{(1-1/\alpha)\beta}d\alpha/\alpha\right).$$

We break the range of integration for α into $[0, 1/10]$ and $[1/10, 1]$. Since

$$\int_{\alpha=1/10}^{1} e^{(1-1/\alpha)\beta}d\alpha/\alpha = O(1),$$

$$O\left(\sqrt{n}\int_{\alpha=1/10}^{1} \ldots\ d\alpha/\alpha\right) = O(\sqrt{n}).$$

On $[0, 1/10]$, $1 - 1/\alpha < -1/2\alpha$, so

$$\sqrt{n}\int_{\alpha=0}^{1/10} e^{(1-1/\alpha)\beta}d\alpha/\alpha = O\left(\sqrt{n}\int_{0}^{1/10} e^{-\beta/2\alpha}\,d\alpha/\alpha\right).$$

Let $\gamma = \beta/(2\alpha)$, so $d\alpha/\alpha = -d\gamma/\gamma$ and

$$\int_{0}^{1/10} e^{-\beta/2\alpha}\,d\alpha/\alpha = \int_{\gamma=5\beta}^{\infty} e^{-\gamma}\,d\gamma/\gamma.$$

Since β is only required to be less than $1/(1+r)$, and $0 < r < 1$, we can presume that $5\beta > 1$. Then, since

$$\int_{1}^{\infty}(e^{-x}/x)dx = O(1),$$

we get

$$\sqrt{n}\int_{\alpha=0}^{1/10} e^{(1-1/\alpha)\beta}d\alpha/\alpha = O(\sqrt{n}).$$

Hence $|g(z)| = O(\sqrt{n})$ regardless of where z lies on $|z| = r$. □

3. Inclusion of Specific Terms in TGFs

Recall that $f(k_1, \ldots, k_m)$ is the length of a shortest TGF which contains at least one permutation of the positive integer sequence $(k_1, \ldots, k_m)$. If there is no such TGF, we say that $f(k_1, \ldots, k_m)$ is undefined.

Theorem 3: $f(k_1, \ldots, k_m)$ is always defined for $m \le 4$ but can be undefined for $m \ge 5$.

Proof: Let $k = \max\{k_1, k_2, k_3, k_4\}$ and assume with no loss in generality that $k_1 \le k_2$ and $k_3 \le k_4$. Then $k_2k_11 \ldots 1k_3k_4$ with k 1's in the middle is a TGF. However, $f(4, 5, 6, 7, 8)$ is undefined since, according to the monotonicity property, at least three numbers from $\{4, 5, 6, 7, 8\}$ must appear in increasing order (away from the 1's) on the same side of the block of 1's, and this is clearly impossible for a TGF. □

Theorem 4: $f(k) = \lceil \log_2 k \rceil + 2$ for $k \ge 2$.

Proof: Since the largest possible term in a sequence in T_n is 2^{n-2} (from 11248 ... 2^{n-2}, for example), $f(k) \ge \lceil \log_2 k \rceil + 2$ for $k \ge 2$. Conversely, given $k \ge 2$ and its expansion as a sum of powers of 2, say,

$$k = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_p} \text{ with } 0 \le k_1 < k_2 < \cdots < k_p,$$

let $u_1 < u_2 < \cdots < u_q$ be all integers in $\{0, 1, \ldots, k_p\} \backslash \{k_1, \ldots, k_p\}$. Then the $(k_p + 2)$-term sequence

$$2^{k_p}, \ldots, 2^{k_2}, 2^{k_1}, 1, 2^{u_1}, 2^{u_2}, \ldots, 2^{u_q}$$

is a TGF since each 2^x equals 1 plus all terms 2^y with $y < x$. If $k = 2^{k_p}$, then it follows that

$$f(k) \le k_p + 2 = \log_2 k + 2;$$

if $k > 2^{k_p}$, then the addition of k to the left end of the sequence gives another TGF, from which

$$f(k) \le k_p + 3 \le \lceil \log_2 k \rceil + 2$$

follows. Hence,

$$f(k) = \lceil \log_2 k \rceil + 2 \text{ for } k \ge 2. \quad □$$

The next steps beyond Theorem 4 are to consider $f(k_1, k_2)$ and $f(k_1, k_2) - f(k_2)$ when $k_1 \le k_2$. We have systematically verified

$$f(k_1, k_2) \le f(k_2) + 1 \quad (k_1 \le k_2)$$

for all $k_2 \le 16$, but do not know if this holds for larger k_2. Similarly, we do not know if there is a fixed c such that

$$f(k_1, k_2, k_3) \le f(k_2, k_3) + c \text{ whenever } k_1 \le k_2 \le k_3.$$

However, we do have the following result.

Theorem 5: If $k_1 \le k_2 \le k_3 \le k_4$, then $f(k_1, k_2, k_3, k_4) - f(k_2, k_3, k_4)$ can be arbitrarily large.

The following lemma is used in the proof of Theorem 5. We will prove the lemma shortly. Here, $\lfloor x \rfloor$ is the integer part of x.

Lemma 1: $f(k, k + 1, k + 2, k + 3) \ge \lfloor k/3 \rfloor + 6$ for $k > 3$.

Proof of Theorem 5: Let

$$(k_1, k_2, k_3, k_4) = (k, k+1, k+2, k+3)$$

with $k + 1 = 2^p$ and $p \geq 3$. Then

$$f(k+1, k+2, k+3) \leq p + 5 = \log_2(k+1) + 5$$

since

$$2^p + 2, 2^p + 1, 1, 1, 1, 2, 4, 8, \ldots, 2^p$$

is a TGF in T_{p+5}. When this is combined with the conclusion of Lemma 1, we have

$$f(k_1, k_2, k_3, k_4) - f(k_2, k_3, k_4) \geq \lfloor k/3 \rfloor + 1 - \log_2(k+1),$$

and the right-hand side can be made arbitrarily large. □

Proof of Lemma 1: Let $S = x_1 \ldots x_n$ be a *shortest* TGF that contains the integers in

$$K = \{k, k+1, k+2, k+3\}, k > 3.$$

By the monotonicity property, $x_i \leq k + 3$ for all i.

CLAIM: $K = \{x_1, x_2, x_{n-1}, x_n\}$.

To prove the Claim, note first that since $k > 3$, it is impossible for more than two elements of K to appear in increasing order away from the center on the same side of the sequence 1, 1. Thus, there must be two elements of K on each side of the block of 1's. Since S is a shortest TGF, elements of K should be at the beginning and end of S, and there are no repetitions of elements of K. Thus, x_1 and x_n are in K. The Claim follows by monotonicity of S.

We now use the Claim to analyze the following three cases:

Case 1: $x_1, x_2 = k + 1, k;\ x_{n-1}, x_n = k + 2, k + 3$.

Case 2: $x_1, x_2 = k + 2, k;\ x_{n-1}, x_n = k + 1, k + 3$.

Case 3: $x_1, x_2 = k + 3, k;\ x_{n-1}, x_n = k + 1, k + 2$.

The other three possible cases are symmetric to these.

Case 1: By the construction process, this case forces S to be

$$k+1, k, 1, \ldots, 1, k+2, k+3.$$

By monotonicity, all remaining terms are 1's and so there are $k + 2$ 1's. It follows that $n = (k+2) + 4 = k + 6$, and $k + 6 \geq \lfloor k/3 \rfloor + 6$.

Case 2: For this case, let

$$S = k+2, k, p, \ldots, q, k+1, k+3.$$

To obtain $k + 2$ by the construction process, we must have $p \leq 2$, and similarly, $q \leq 2$. Hence, all terms from p through q are ≤ 2. Since there must be at least two 1's, and since $p + \cdots + q \geq k + 1$, we note that to obtain $k + 1$ by construction, we must have

$$n \geq 2 + \left\lceil \frac{k-1}{2} \right\rceil + 4 = \left\lceil \frac{k-1}{2} \right\rceil + 6 \geq \lfloor k/3 \rfloor + 6.$$

Case 3: Let

$$S = k + 3,\ k,\ p,\ \ldots,\ q,\ k + 1,\ k + 2$$

which forces $q = 1$ and $p \leq 3$. Since the p through q part must end in 111 or 211, and since every other term in this part is ≤ 3 by the monotonicity property,

$$n \geq 3 + \left\lceil \frac{k + 1 - 4}{3} \right\rceil + 4 = \lceil k/3 \rceil + 6 \geq \lfloor k/3 \rfloor + 6. \ \square$$

4. Remarks

Questions of uniqueness in finite measurement structures are proving to be a rich source of interesting combinatorial and number-theoretic problems, as shown in [2, 3] and the present paper, and summarized in [4, 8]. Our story here is the familiar one of encountering Fibonacci-like structures in an area where none was visible at the start. Not only are TGFs natural generalizations of the basic Fibonacci sequence in their two-sidedness and their relaxation of the requirement that a new addition be the sum of exactly two neighbors, but the sequence $t_1, t_2, t_3, \ldots$ that counts the number of TGFs has a recurrence in which the next term is determined by precisely its two immediate predecessors.

The most obvious problems left open in the paper concern boundedness, or better, of $f(k_1, k_2, k_3) - f(k_2, k_3)$ and $f(k_1, k_2) - f(k_2)$ when $k_1 \leq k_2 \leq k_3$. A further possibility for investigation is $f^*(k_1, \ldots, k_m)$, the length of the shortest TGF, if any, that has $k_1, \ldots, k_m$ as a subsequence.

We mention two generalizations of two-sided generalized Fibonacci sequences. The first is also two-sided and is constructed like a TGF except that the value of a new term at either end can equal the sum of one or more contiguous terms (including a single 1) located anywhere in the sequence constructed thus far. Some results for this generalization are included in [2].

The other generalization is one of a large number of things that might be referred to as generalized Fibonacci trees. The tree we have in mind is constructed like a TGF except that it has N rather than 2 branches extending away from a root that consists of two 1's. The value of a new term added to a branch is the sum of one or more extant terms consisting of either (a) immediate predecessors on that branch, or (b) all those predecessors plus one or both root 1's, or (c) all its branch predecessors plus both root 1's plus terms contiguous to the root along some other branch. We are not aware of results for this generalization.

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