EQUIVALENCE OF PIZA'S PRIMALITY CRITERION WITH THAT OF GOULD-GREIG AND ITS DUAL RELATIONSHIP TO THE MANN-SHANKS CRITERION

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1. Introduction

The famous amateur mathematician Pedro A. Piza (1896-1956), who spent most of his life in business in Puerto Rico, discovered many interesting things in geometry and number theory. In his paper on Fermat coefficients [5], he discovered a criterion for primality which runs as follows:

$$2n + 1 \text{ is prime iff } k \,\Big|\, \binom{2n - k}{k - 1} \text{ for all } k,\ 1 \le k \le n. \tag{1.1}$$

Actually, he left an ellipsis in the proof. He said that when k is composite the proof that k divides the binomial coefficient when $2n + 1$ is prime is somewhat more complicated but not difficult and he left this to the interested reader.

Eighteen years later, Henry Mann and Daniel Shanks [4] discovered another attractive primality criterion which may be stated as follows:

$$C \ge 2 \text{ is prime iff } R \,\Big|\, \binom{R}{C - 2R} \text{ for every } R \text{ such that}$$
$$C/3 \le R \le C/2,\ R \ge 1. \tag{1.2}$$

In their criterion, the binomial coefficients are arranged in an array where each row is shifted two units over from the preceding. Then the criterion may be stated more pithily in the following way: A column number C is prime if and only if every binomial coefficient in that column is divisible by its corresponding row number R. (See Table 3 below.)

Then Gould and Greig [3] obtained a primality criterion using a "Lucas" triangle. Diagonals in this triangle sum to Lucas numbers, whereas in the usual Pascal triangle they sum to Fibonacci numbers. The criterion runs as follows:

$$D \ge 2 \text{ is prime iff } D \,|\, A(D - j, j) \text{ for all } j,\ 1 \le j \le D/2, \tag{1.3}$$

where

$$A(n, k) = \binom{n}{k} + \binom{n - 1}{k - 1}.$$

It was shown in [3] that this can be reformulated in the equivalent form:

$$C \ge 2 \text{ is prime iff } R \,\Big|\, \binom{-R}{C - 2R} \text{ for all } R,\ 1 \le R \le C/2. \tag{1.4}$$

This made the criterion dual to that of Mann and Shanks, since only a minus sign is different in comparison to (1.2), but of course the coefficients differ.

We shall show here that Piza's criterion may be reformulated as follows:

$$C \geq 2 \text{ is prime iff } R \,\Big|\, (-1)^C \binom{-R}{C - 2R} \text{ for all } R,\ 1 \leq R \leq C/2. \tag{1.5}$$

Since the sign $(-1)^C$ does not affect divisibility, it follows that Piza's criterion is equivalent to that of Gould-Greig. Table 4 below shows Piza's coefficients.

All of these criteria are susceptible to extensions to generalized binomial coefficients—such as Gaussian or q-coefficients, Fibonomial coefficients, s-Fibonomial coefficients, etc.—as was shown for the Mann-Shanks criterion in [1] and [2]. The Mann-Shanks criterion, requiring fewer divisibility tests, is more efficient than the criteria of Piza or Gould-Greig.

2. Proofs and Discussion

Let us first examine Piza's array from (1.1). We have the following display of $\binom{2n-k}{k-1}$:

TABLE 1. Piza Array

n	$2n+1$	1	2	3	4	5	6	7 ... k
1	3	1						
2	5	1	2					
3	7	1	4	3				
4	(9)	1	6	10	4			
5	11	1	8	21	20	5		
6	(13)	1	10	36	56	35	6	
7	15	1	12	55	120	126	56	7

Thus 9 is not prime since $3 \nmid 10$; 15 is not prime since $3 \nmid 55$, $5 \nmid 126$, $6 \nmid 56$. We may rearrange the table to make it look more like the usual Pascal array of $\binom{n}{k}$:

TABLE 2. Modified Piza Array

n	$n+2$	0	1	2	3	4	5	6 ... k
1	3	1	1					
2	4	1	2	1				
3	5	1	3	3	1			
4	6	1	4	6	4	1		
5	7	1	5	10	10	5	1	
6	8	1	6	15	20	15	6	1
7	(9)	1	7	21	35	35	21	7
8	10	1	8	28	56	70	56	28
9	11	1	9	36	84	126	126	84
10	12	1	10	45	120	210	252	210
11	(13)	1	11	55	165	330	462	462
12	14	1	12	66	220	495	792	924

Examination of these arrays shows that Piza's criterion may be reformulated as follows:

$$n + 2 \text{ is prime iff } k + 1 \left| \binom{n - k}{k} \right. \text{ for all } k,\ 0 \le k \le n/2. \tag{2.1}$$

Make the replacement $n \leftarrow n - 2$ and this becomes

$$n \text{ is prime iff } k + 1 \left| \binom{n - 2 - k}{k} \right. \text{ for all } k,\ 0 \le k \le n/2 - 1. \tag{2.2}$$

Then make the replacement $k \leftarrow k - 1$ and this, in turn, becomes

$$n \text{ is prime iff } k \left| \binom{n - k - 1}{k - 1} \right. \text{ for all } k,\ 1 \le k \le n/2. \tag{2.3}$$

However,

$$\binom{n - k - 1}{k - 1} = \binom{n - k - 1}{n - 2k} = (-1)^n \binom{-k}{n - 2k},$$

so that Piza's criterion takes the form

$$n \text{ is prime iff } k \left| (-1)^n \binom{-k}{n - 2k} \right. \text{ for all } k,\ 1 \le k \le n/2, \tag{2.4}$$

which is what we asserted in (1.5). The $(-1)^n$ may be dropped, as we said, so that Piza's criterion is equivalent to the Gould-Greig criterion.

Now let us return to the original (1.1) and make the replacement $k \leftarrow n - k$. We obtain the equivalent form of Piza's criterion:

$$2n + 1 \text{ is prime iff } n - k \left| \binom{n + k}{n - k - 1} \right. \text{ for all } k,\ 0 \le k \le n - 1. \tag{2.5}$$

Since $\binom{n+k}{n-k-1} = \binom{n+k}{2k+1}$, we may restate (2.5) as:

$$2n + 1 \text{ is prime iff } n - k \left| \binom{n + k}{2k + 1} \right. \text{ for all } k,\ 0 \le k \le n - 1. \tag{2.6}$$

This is an interesting variant because in [1] it was shown that the Mann-Shanks criterion could be rephrased in the form:

$$2n + 1 \text{ is prime iff } n - k \left| \binom{n - k}{2k + 1} \right. \text{ for all } k,\ 0 \le k \le \frac{n - 1}{3}. \tag{2.7}$$

Thus, the Piza criterion is a kind of dual to that of Mann-Shanks in that one has $n + k$ and the other has $n - k$.

We close by setting down the original Mann-Shanks array (1.2) followed by the array of Piza-Gould-Greig in the form (1.5):

TABLE 3. Mann-Shanks Array

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19 ... C
1	1	1																
2			1	2	1													
3					1	3	3	1										
4							1	4	6	4	1							
5									1	5	10	10	5	1				
6											1	6	15	20	15	6	1	
7													1	7	21	35	35	21
8															1	8	28	56
9																	1	9
⋮																		
R																		

TABLE 4. Piza-Gould-Greig Array

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	... C
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
2			1	2	3	4	5	6	7	8	9	10	11	12	
3					1	3	6	10	15	21	28	36	45	55	
4							1	4	10	20	35	56	84	120	
5									1	5	15	35	70	126	
6											1	6	21	56	
7													1	7	
⋮															
R															

Mann-Shanks is far more efficient, requiring fewer divisibility tests in a column. In Table 4 we must test each R for $1 \le R \le C/2$, whereas in Table 3 we test only values of R with $C/3 \le R \le C/2$.

The multiple charms of Pascal's triangle are far from exhausted.

References

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