GROUPS OF INTEGRAL TRIANGLES

Ernest J. Eckert
University of South Carolina at Aiken, SC 29801

Preben Dahl Vestergaard
Institute of Electronic Systems, Aalborg University, Denmark

(Submitted October 1987)

The Group H_γ. An *integral* (*rational*) triangle $T = (a, b, c)$ is a triangle having integral (rational) side lengths a, b, c. Two rational triangles

$$T_1 = (a_1, b_1, c_1) \quad \text{and} \quad T_2 = (a_2, b_2, c_2)$$

are *equivalent* if one is a rational multiple of the other:

$$(a_2, b_2, c_2) = (ra_1, rb_1, rc_1) \text{ for rational } r.$$

As our favorite representative for a class of equivalent rational triangles, we take the *primitive* integral triangle (a, b, c) in which a, b, c have no common factor greater than 1. That is, for any positive rational number r we identify (ra, rb, rc) with (a, b, c).

A *pythagorean* triangle is an integral triangle (a, b, c) in which the angle γ, opposite side c, is a right angle. Equivalently, (a, b, c) is a pythagorean triangle if a, b, c satisfy the pythagorean equation

$$a^2 + b^2 = c^2. \tag{1}$$

An angle β, $0 < \beta < \pi$, is said to be *pythagorean* if β, or $\pi - \beta$, is an angle in some pythagorean triangle or, equivalently, if it has rational sine and cosine. A *heronian* triangle is an integral triangle with rational area. Clearly, an integral triangle is heronian if and only if each of its angles α, β, γ, is pythagorean.

In [1] the set of primitive pythagorean triangles is made into a group $H_{\pi/2}$. The group operation is, basically, *addition of angles modulo* $\pi/2$. When placed in *standard position* (Fig. 1), a primitive pythagorean triangle $T = (a, b, c)$ is uniquely determined by the point P on the unit circle,

$$P = (\cos\beta, \sin\beta) = (a/c, b/c).$$

Geometrically, the sum of two such triangles, (a, b, c) and (A, B, C), is obtained by adding their central angles β_1 and β_2. If $\beta_1 + \beta_2$ equals or exceeds $\pi/2$, then the angle sum is reduced modulo $\pi/2$. The identity element is the (degenerate) triangle $(1, 0, 1)$ with $\beta = 0$. The inverse of $T = (a, b, c)$ is $-T = (b, a, c)$. Thus, in $H_{\pi/2}$ we must distinguish between (a, b, c) and (b, a, c), even though they are congruent triangles. Analytically, the sum of (a, b, c) and (A, B, C) may be expressed

$$(a, b, c) + (A, B, C) = \begin{cases} (aA - bB, bA + aB, cC) & \text{when } aA - bB > 0, \\ (bA + aB, bB - aA, cC) & \text{when } aA - bB \le 0. \end{cases} \tag{2}$$

With this definition of the sum of two triangles, $H_{\pi/2}$ becomes a *free abelian group*. The set of generators of $H_{\pi/2}$ may be taken to be the set of triangles $T_p = (r, s, p)$ with p prime, $p \equiv 1 \pmod 4$, and $r > s$ ([1, p. 25]). Thus, any primitive pythagorean triangle can be written as a unique linear combination of the generators with integral coefficients.

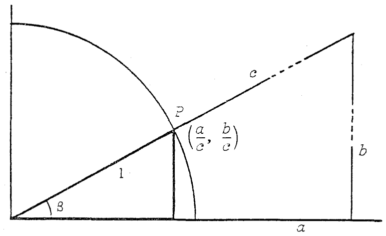

FIGURE 1

The set of pythagorean triangles may be characterized as that subset of the integral triangles for which $\gamma = \pi/2$. One may ask if the requirement that γ be a right angle is essential for defining a group structure. The answer is no. Let us fix an angle γ, $0 < \gamma < \pi$, with rational cosine, $\cos \gamma = u/w$, and denote by H_γ that subset of the integral triangles having one fixed angle γ. The condition for a triangle $T = (a, b, c)$ to belong to H_γ is that a, b, and c satisfy the generalized pythagorean theorem (the law of cosine)

$$c^2 = a^2 + b^2 - 2abu/w. \tag{3}$$

To define addition of two triangles in H_γ, we proceed almost as we did for the case of $H_{\pi/2}$ above, guided by our geometric intuition. When placed in *standard position* (Fig. 2), a primitive γ-angled triangle $T = (a, b, c)$ is uniquely determined by its central angle β or, equivalently, by the point P on the unit circle. Geometrically, the sum of two triangles (a, b, c) and (A, B, C) in H_γ is obtained by adding their two central angles β_1 and β_2. If $\beta_1 + \beta_2$ equals or exceeds $(\pi - \gamma)$, then the angle sum is reduced modulo $(\pi - \gamma)$. The identity element is the (degenerate) triangle $(1, 0, 1)$ with $\beta = 0$. The inverse of $T = (a, b, c)$ is $-T = (b, a, c)$. Thus, in H_γ we must distinguish between (a, b, c) and (b, a, c) even though they are congruent triangles. Analytically, the sum of (a, b, c) and (A, B, C) may be expressed

$$(a, b, c) + (A, B, C) = \begin{cases} (aA - bB,\ aB + bA - 2bBu/w,\ cC) & \text{if } aA - bB > 0, \\ (aB + bA - 2aAu/w,\ bB - aA,\ cC) & \text{if } aA - bB \le 0. \end{cases} \tag{4}$$

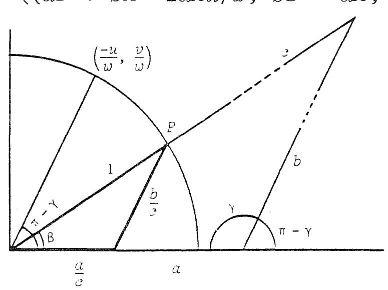

FIGURE 2

To see that (4) defines a group operation on H_γ, we must show that

$$T = (a, b, c) + (A, B, C)$$

satisfies (3). This is not very difficult but somewhat tedious. Since addition of γ-angled triangles corresponds to addition of their central angles, the operation is associative and commutative. Simple computations will show that $(1, 0, 1)$ is the identity element, and that $-(a, b, c) = (b, a, c)$. We note that (4) reduces to (2) when $\gamma = \pi/2$.

Using the angle γ for which $\cos \gamma = -5/13$, we give a few examples of addition of triangles as defined by (4):

(13, 11, 20) + (119, 65, 156) = (4, 13, 15),
(13, 11, 20) + (65, 119, 156) = (182, 29, 195),
2(11, 13, 20) = (11, 13, 20) + (11, 13, 20) = (308, 39, 325),
3(11, 13, 20) = (2881, 4823, 6500).

The sum of two triangles is required to be a primitive member of H_γ, so cancellation of common factors of the three coordinates on the right of (4) may be necessary. To obtain integral components, multiplication of the three coordinates by $w = 13$ may be necessary. For computational purposes, it is worth noting, from (4), that if a triple $T = (a, b, c)$ appears with $a \le 0$, then

$$T = (b - 2au/w, -a, c)$$

after the required reduction of the central angle modulo $(\pi - \gamma)$. We summarize in Proposition 1.

Proposition 1: The set H_γ of primitive γ-angled triangles is an abelian group under the operation, called addition, defined by (4). The identity element in H_γ is (1, 0, 1), and the inverse of (a, b, c) is (b, a, c).

The group $H_{\pi/2}$ is a free abelian group. For values of γ other than $\pi/2$, a characterization of the group H_γ is not so simple. One difficulty is that we have no easy way, so far, of identifying the members of H_γ. For $H_{\pi/2}$, it is well known [2] that a member (r, s, t), i.e., a primitive pythagorean triangle, is generated by a pair of positive integers (m, n), $m > n$ that are relatively prime and have $m + n \equiv 1 \pmod 2$. The generation process is:

$$r = m^2 - n^2,\ s = 2mn,\ t = m^2 + n^2 \quad \text{or} \tag{5}$$

$$r = 2mn,\ s = m^2 - n^2,\ t = m^2 + n^2. \tag{5'}$$

In fact, (5) establishes a one-to-one correspondence between the set of all such pairs (m, n) and the set of all primitive pythagorean triangles (r, s, t) with odd r. The pair (m, n) is called the *generator* for (r, s, t).

To obtain a generating process for γ-angled triangles, akin to (5) and (5′) for pythagorean triangles, we shall make the restriction that $\cos\gamma = u/w$, $\sin\gamma = v/w$ are both rational numbers. To simplify the derivation of the process, which is geometrically inspired, we also make the assumption that $\pi > \gamma > \pi/2$, so that $u/w < 0$, i.e., $u < 0$. Thus, $(|u|, v, w)$ is a pythagorean triangle, and $(\pi - \gamma)$ is a pythagorean angle, $(\pi - \gamma) < \pi/2$. The central angle of any triangle T in H_γ is then also pythagorean, and T is a heronian triangle.

With each pythagorean angle β, $0 < \beta < (\pi - \gamma)$, we can associate a unique primitive pythagorean triangle (r, s, t) having central angle β, and also a unique primitive γ-angled triangle (a, b, c) having central angle β (Fig. 3). It follows that there is a one-to-one mapping ϕ from the subset of $H_{\pi/2}$ having central angle $\beta < (\pi - \gamma)$ onto H . Using only elementary geometry (Fig. 3), one may show that ϕ and ϕ^{-1} can be represented by matrices in the following way.

$$(a, b, c) = \phi(r, s, t) = \begin{bmatrix} v & u & 0 \\ 0 & w & 0 \\ 0 & 0 & v \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = (vr + us, ws, vt), \tag{6}$$

$$(r, s, t) = \phi^{-1}(a, b, c) = \begin{bmatrix} w & -u & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = (wa - ub, vb, wc). \tag{7}$$

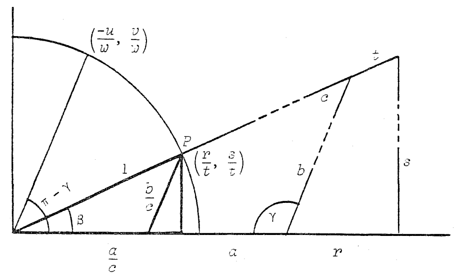

FIGURE 3

$$\frac{a}{c} = \frac{r}{t} - \frac{s}{t}\cos(\pi - \gamma), \quad \frac{b}{c} = \frac{s}{t}\csc(\pi - \gamma),$$

$$(a, b, c) - (vr + us, ws, vt)$$

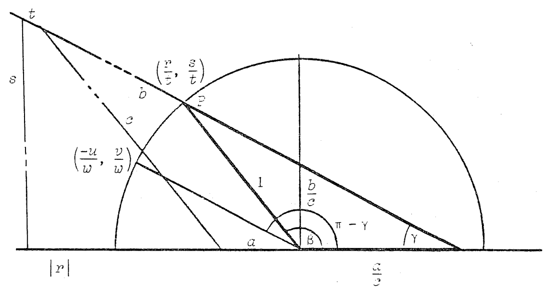

FIGURE 4

$$\frac{a}{c} = \frac{r}{t} - \frac{s}{t}\cot(\pi - \gamma), \quad \frac{b}{c} = \frac{s}{t}\csc(\pi - \gamma).$$

$$(a, b, c) = (vr + us, ws, vt)$$

Cancellations of common factors of the three components on the right of (6) and (7) may be necessary in order to arrive at primitive triangles. We give a few examples to illustrate the use of (6). As above, γ is defined by

$$\cos\gamma = -5/13 = u/w, \ \sin\gamma = 12/13 = v/w.$$

$$\phi(4, 3, 5) = \begin{bmatrix} 12 & -5 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 12 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = (33, 39, 60) = (11, 13, 20).$$

$$\phi(3, 4, 5) = (4, 13, 15). \qquad \phi(35, 12, 37) = (30, 13, 37).$$

Equation (6) makes it possible to construct members of H_γ from members of $H_{\pi/2}$. The generating process for γ-angled triangles is at hand: substitute (5) and (5′) into (6) and remember that the central angle β must satisfy $\beta < (\pi - \gamma)$. Thus, when using (5), we must have

$$\tan\frac{\beta}{2} = \frac{n}{m} < \tan\frac{\pi - \gamma}{2} = \frac{w + u}{v},$$

and, when using (5′), we must have

$$\tan\frac{\beta}{2} = \frac{m - n}{m + n} < \frac{w + u}{v}, \text{ so that } \frac{n}{m} > \frac{w - u - v}{w - u + v} = \frac{v - w}{u}.$$

Summarizing, we formulate the generating process for γ-angled triangles in Proposition 2.

Proposition 2: For a given angle γ, $\pi/2 < \gamma < \pi$, $(\cos\gamma, \sin\gamma) = (u/w, v/w)$, a γ-angled triangle (a, b, c) is generated by means of:

$$a = v(m^2 - n^2) + 2umn, \quad b = 2wmn, \quad c = v(m^2 + n^2) \quad \text{or} \tag{8}$$

$$a = 2vmn + u(m^2 - n^2), \quad b = w(m^2 - n^2), \quad c = v(m^2 + n^2). \tag{8'}$$

Here m and n are relatively prime positive integers, $m > n$ and $m + n = 1$ (mod 2). Furthermore, for (8), (m, n) must satisfy $n/m < (w + u)/v$, and for (8′), $n/m > u/(w + v)$. Each γ-angled triangle is obtained in this way.

We illustrate the use of (8) by a few examples. As before, γ is given by $(\cos\gamma, \sin\gamma) = (-5/13, 12/13) = (u/w, v/w)$.

$(m, n) = (2, 1)$. $\quad (a, b, c) = (16, 52, 60) = (4, 13, 15)$.
$(m, n) = (3, 2)$. $\quad (a, b, c) = (0, 1, 1) = (1, 0, 1)$.
$(m, n) = (4, 1)$. $\quad (a, b, c) = (35, 26, 51)$.
$(m, n) = (5, 2)$. $\quad (a, b, c) = (38, 65, 77)$.

If $(u, v, w) = (0, 1, 1)$, so that $\gamma = \pi/2$, then (8) and (8′) reduce to (5) and (5′). Thus, Proposition 2 is a generalization of the euclidean process of generating pythagorean triangles.

In the case of $0 < \gamma < \pi/2$, not covered by Proposition 2, the derivation of the generation process of γ-angled triangles is only slightly more complicated. The key step, however, is still the formal substitution of (5), (5′) into (6) with a slight modification. If $0 < \beta < \pi/2$, so that $P = (r/t, s/t)$ is in the first quadrant, we simply substitute (5) or (5′) into (6) depending on whether r is odd or even. For $\beta = \pi/2$, $P = (0/1, 1/1)$, we use $(m, n) = (1, 1)$ to generate the pythagorean triple $(0, 1, 1)$, from which we obtain

$$\phi(0, 1, 1) = (u, w, v).$$

If $\pi/2 < \beta < (\pi - \gamma)$, so that P is in the second quadrant (Fig. 4), we write

$$P = \left(\frac{-r}{t}, \frac{s}{t}\right), \quad r > 0,$$

and consider the point $P' = (r/t, s/t)$ in the first quadrant. The corresponding angle β', $(\cos\beta', \sin\beta') = (r/t, s/t)$, must satisfy $\gamma < \beta' < \pi/2$. We then apply ϕ to $(-r, s, t)$ to arrive at the member of H_γ that corresponds to P. We summarize in Proposition 2′.

Proposition 2′: For a given angle γ, $0 < \gamma < \pi/2$, $(\cos\gamma, \sin\gamma) = (u/w, v/w)$, a γ-angled triangle (a, b, c) is generated by means of:

$$a = v(m^2 - n^2) + 2umn, \quad b = 2wmn, \quad c = v(m^2 + n^2) \text{ or} \tag{8}$$

$$a = 2vmn + u(m^2 - n^2), \quad b = w(m^2 - n^2), \quad c = v(m^2 + n^2) \text{ or} \tag{8'}$$

$$a = -v(m^2 - n^2) + 2umn, \quad b = 2wmn, \quad c = v(m^2 + n^2) \text{ or} \tag{9}$$

$$a = -2vmn + u(m^2 - n^2), \quad b = w(m^2 - n^2), \quad c = v(m^2 + n^2). \tag{9'}$$

Here m and n are relatively prime positive integers, $m > n$ and $m + n \equiv 1$ (mod 2). Furthermore, for (9), (m, n) must satisfy

$$\frac{n}{m} > \frac{v}{w + u},$$

and for (9′),

$$\frac{n}{m} < \frac{u}{w + v} .$$

Each γ-angled triangle is obtained in this way except (u, w, v), which is obtained from (8) by taking $(m, n) = (1, 1)$.

H_0 and H_π. It is tempting to consider the two limiting cases: $\gamma = 0, \pi$. Since then the triangles collapse, we shall call (a, b, c) a *triple*. But the rule (4) for adding two such triples makes sense and, in fact, defines a group operation on H_γ, $\gamma = 0, \pi$.

For $\gamma = 0$, we have $\cos \gamma = 1 = u/w$, $\sin \gamma = 0 = v/w$, and we may take

$$(u, v, w) = (1, 0, 1).$$

The condition for the triple (a, b, c) to belong to H_0 is:

$$c^2 = a^2 + b^2 - 2ab, \text{ i.e., } c = |a - b|. \tag{10}$$

Similarly, the condition for the triple (a, b, c) to belong to H_π is:

$$c^2 = a^2 + b^2 + 2ab, \text{ i.e., } c = a + b. \tag{11}$$

Addition of two triples in H_0 and H_π is defined by rewriting (4) with $\cos \gamma = 1$ and $\cos \gamma = -1$, respectively.

For (a, b, c) and (A, B, C) in H_0:

$$(a, b, c) + (A, B, C) = \begin{cases} (aA - bB,\ aB + bA - 2bB,\ cC) & \text{when } aA - bB > 0, \\ (aB + bA - 2aA,\ bB - aA,\ cC) & \text{when } aA - bB \le 0. \end{cases}$$

For (a, b, c) and (A, B, C) in H_π:

$$(a, b, c) + (A, B, C) = \begin{cases} (aA - bB,\ aB + bA + 2bB,\ cC) & \text{when } aA - bB > 0, \\ (aB + bA + 2aA,\ bB - aA,\ cC) & \text{when } aA - bB \le 0. \end{cases}$$

It is straightforward to verify that H_0 and H_π become groups under these operations. Note that the generation process described above makes no sense for triples in H_0 and H_π. Fortunately, (10) and (11) already provide easy methods of constructing members of H_0 and H_π.

Open Problems. How does the group structure of H_γ vary as the parameter γ runs through the pythagorean angles from 0 to π? For each such γ, the group operation is defined by (4), but the group structures are, in general, different. For example, $H_{\pi/2}$ has no nontrivial element of finite order, whereas H_π has no element of infinite order. For other values of γ, H_γ has nontrivial elements of finite order as well as elements of infinite order. A description of the isomorphism classes of the family of groups H_γ, $0 \le \gamma \le \pi$, is desirable. Another question is: For which angles γ is H_γ isomorphic with $H_{\pi/2}$? More generally, for which angles γ_1 and γ_2 are H_{γ_1} and H_{γ_2} isomorphic? If $H_{\pi/2}$ and H_γ are isomorphic, which properties of pythagorean triangles can be transferred to γ-angled triangles? For example, the number of primitive pythagorean triangles (r, s, t) having the same hypotenuse t may be determined by using the group structure of the group $H_{\pi/2}$, see [1]. If (a, b, c) is in H_γ and H_γ is isomorphic with $H_{\pi/2}$, can one determine the number of γ-angled triangles having the same "hypotenuse" c?

A great many papers about pythagorean triangles have appeared in the literature presenting various properties of these beautiful triangles. See, for example [2]. Very possibly some properties of pythagorean triangles can be generalized so as to be applicable to γ-angled triangles. How exclusive is the requirement that γ be a right angle?

Acknowledgment

The authors wish to express their gratitude to the referee for making corrections, suggesting improvements in style, and finding a counterexample to a conjecture we had made.

References

1. Ernest J. Eckert. "The Group of Primitive Pythagorean Triangles." *Math. Mag.* 57 (1984):22-27.
2. Waclaw Sierpinski. *Pythagorean Triangles*. The Scripta Mathematica Studies, No. 9. New York: Yeshiva University, 1962.

*****