CHARACTERIZATIONS OF THREE TYPES OF COMPLETENESS

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Introduction

A sequence is *complete* if every positive integer is a sum of distinct terms of the sequence [1, 3]. In this paper I discuss and characterize this definition and two definitions that generalize it.

In Section 1, I give several examples of complete sequences. Section 2 describes how a theorem due to Brown & Weiss [1] can be used to characterize the complete sequences. In Section 3, *weak completeness* [3] is defined, a sufficient condition for a sequence to be weakly complete is given and, finally, a condition equivalent to weak completeness is presented.

In Section 4, the concept of *completability* is introduced. Several conditions which imply the completability of a sequence are described. A theorem characterizing the completable sequences is proved, and it is used to find an infinite noncompletable sequence. The relations between the concepts of "completeness" discussed are described.

1. Sequences and Completeness

A sequence is a collection of numbers in one-to-one correspondence with the positive integers. Since only sequences of nonnegative integers are considered in this paper, the word "number" will be understood to refer to a nonnegative integer, and the word "sequence" will refer only to sequences of such numbers.

Definition 1: A sequence f is *complete* [3] if every natural number is a sum of one or more distinct terms of the sequence.

Erdös & Graham [2] mean, by a *complete* sequence, a sequence such that every sufficiently large natural number is a sum of distinct terms of the sequence. We will not use "complete" in this sense.

Clearly, the sequence $\{n\} = \{1, 2, 3, 4, 5, \ldots\}$ is complete. However, there exist infinitely many other complete sequences. For example, the sequence $\{1, 2, 3, 4, 8, 12, 16, 20, 24, 28, \ldots\}$ is complete. This follows from our ability to represent each positive integer in mod 4. A similar sequence may be obtained from any number $m > 1$, by appending the numbers from 1 to $m - 1$ to the multiples of m. As we see in the following example, any sequence constructed in a similar manner is complete.

Example 1: Let m be a natural number. Then the sequence f, where

$$f(n) = \begin{cases} (n - m + 1)m, & \text{if } n > m, \\ n, & \text{if } 1 \le n \le m \end{cases}$$

is complete.

Proof: Let n be a natural number, and let r be its least residue mod m. If $r = 0$, then n is a term of f. If $r \neq 0$, then $n - r$ is a multiple of m. If $n - r = 0$, then, once again, n is a term of f; otherwise, n is a sum of distinct terms of f, namely $n - r$ and r.

The Fibonacci sequence $\{1, 1, 2, 3, 5, 8, \ldots\} = f_{1,1}$ is an example of a complete sequence [3]. In this sequence,

$$f_{1,1}(1) = 1, \quad f_{1,1}(2) = 1, \quad \text{and}$$
$$f_{1,1}(n) = f_{1,1}(n-1) + f_{1,1}(n-2) \quad \text{if } n \geq 3.$$

Consider the class of all sequences f that satisfy the recurrence relation

$$f(n) = f(n-1) + f(n-2) \quad \text{if } n \geq 3. \tag{R}$$

The sequences in this class have only two degrees of freedom, since, given the first two terms, the recurrence relation (R) determines all remaining terms. Any ordered pair of whole numbers can be the first two terms of a sequence satisfying (R). The class of these sequences is countably infinite, but any illusions we might have that an infinite number of them are complete are shattered by Proposition 1 which follows. But first, a definition:

Definition 2: Suppose $f_{i,j}(1) = i$, $f_{i,j}(2) = j$, and that $f_{i,j}$ satisfies (R). Then $f_{i,j}$ is called *the Fibonacci sequence beginning with i and j.*

Proposition 1: The Fibonacci sequence beginning with i and j is complete if and only if (i, j) is one of the pairs $(0, 1)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(2, 1)$.

Proof: ("If" part.) Parallels exactly the proof that $f_{1,1}$ is complete.

("Only if" part.) Let f be the Fibonacci sequence beginning with i and j. Suppose f is complete. It is easily seen that 1 must be one of the first two terms of f. If $i = 1$ and $j > 2$, then 2 is not a sum of distinct terms of f. If $j = 1$ and $i > 2$, then 2 is, again, not a sum. So if f is complete, then (i, j) is one of the pairs $(0, 1)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(2, 1)$.

In the next section, we shall derive a characterization of the complete sequences.

2. Brown's Criterion and Its Use in Characterizing All Complete Sequences

Of the three sequences $\{1^{n-1}\}$, $\{2^{n-1}\}$, and $\{3^{n-1}\}$, the first two are complete, and the third is not. the following relations are true for all natural numbers n:

$$1^n < 1 + \sum_{i=1}^{n} 1^{i-1}, \quad 2^n = 1 + \sum_{i=1}^{n} 2^{i-1}, \quad \text{and} \quad 3^n > 1 + \sum_{i=1}^{n} 3^{i-1}.$$

These data suggest that a sequence f may be complete iff, for all $p \geq 1$,

$$f(p+1) \leq 1 + \sum_{i=1}^{p} f(i).$$

A counterexample shows that this is not so: If f is the complete sequence $\{8, 4, 2, 1, 16, 32, 64, 128, \ldots, 2^{n-1}, \ldots\}$, then the inequality is false for some p. For instance, $f(2) = 4$, even though $1 + f(1) = 2$. The important difference between the sequences $\{1^{n-1}\}$, $\{2^{n-1}\}$, and $\{3^{n-1}\}$, and the sequence $\{8, 4, 2, 1, 16, \ldots\}$ is that the first three sequences are nondecreasing, while the fourth is not. The following theorem about nondecreasing sequences with first term 1 can be used to characterize all complete sequences. The theorem is known as "Brown's criterion" since it was first proved by Brown & Weiss [1].

Brown's Criterion: If f is a nondecreasing sequence, and if $f(1) = 1$, then f is complete iff, for all $p \geq 1$,

$$f(p + 1) \leq 1 + \sum_{i=1}^{p} f(i).$$

Let f be any sequence. If f is finite, then f is not complete. If f is infinite but contains no 1, then it is not complete, since 1 is not representable. If f is infinite and contains a 1, then it is either nondecreasing or not. Suppose f is not nondecreasing. Either there is a term that occurs infinitely often in the sequence, or there is not. If there is not, then, without affecting its completeness, the terms of the sequence can be rearranged so that it is nondecreasing. Suppose there is a term of the sequence f that repeats infinitely often. The following theorem will show that there is a nondecreasing sequence g that is complete if and only if f is.

Theorem 1: Let the sequence f contain a term which is repeated infinitely often. Then there is a sequence g that is nondecreasing and which is complete if and only if f is.

Proof (and construction): Suppose the value of the least term, in magnitude, that repeats infinitely often is k. If there is no term of f greater than k, then the terms of f less than k can be reordered, and the term k (infinitely repeated) tacked on to the end, to obtain the sequence g. By this procedure, $\{5, 4, 3, 2, 6, 6, 1, 6, \ldots\}$ can be turned into $\{1, 2, 3, 4, 5, 6, 6, \ldots\}$.

If there are terms greater than k, then we show that the removal of all terms of f that are greater than k will not affect its completeness. First, note that the removal of terms from a sequence that is not complete cannot render the sequence complete; so all that must be proved is that, if the sequence f is complete prior to the removal of all terms greater than k, it will remain complete.

Suppose f is complete and all such terms are removed. Let n be a natural number. If $n \leq k$, then n is a sum of distinct terms of the original sequence none of which is greater than k, so it is a sum of distinct terms of the new sequence. If n is greater than k, then n is the sum of a multiple of k and a nonnegative integer less than k, that is,

$$n = ak + r, \text{ where } 0 \leq r < k.$$

If $r = 0$, then, since k is infinitely repeated, n is a sum of distinct terms of the new sequence. If $r \neq 0$, then ak is the sum of distinct terms k, while r is a sum of distinct terms all less than k. So n is the sum of distinct terms of the new sequence. The cases have been exhausted; thus, the new sequence is complete if the original sequence is complete.

Hence, there is no loss of generality in assuming that no term of f is greater than k, since all such terms can be dropped, and the resulting sequence can be reordered into a nondecreasing sequence, as described above.

So we may assume, without loss of generality, that f is nondecreasing. If f contains zeros, they can be removed, again without affecting completeness, so assume f contains no zeros. Brown's criterion may immediately be applied to decide whether f is complete—for, since f contains a 1 but no zeros, $f(1)$ must be 1.

Briefly, then, the procedure for testing a sequence f for completeness is as follows:

i) If f is finite, or if f contains no 1, then f is not complete.

ii) If some number occurs infinitely often in the sequence f, then remove all terms of f that are greater than the least term so repeated, if any terms greater than the least term do exist.

iii) If f is not nondecreasing, then reorder it so that it is. Do not remove any nonzero terms of the sequence to accomplish this.

iv) If f contains any zeros, remove them, since a sum of distinct integers containing zeros clearly is still a sum of distinct integers.

v) Prove or disprove that the inequality

$$f(p + 1) \leq 1 + \sum_{i=1}^{p} f(i)$$

holds for all $p \geq 1$.

The complete sequences have been characterized!

The limitation of completeness, as a mathematical statement of the intuitive idea of the "richness" of a sequence, is not one of undue generality but, rather, is a failure to include sequences which are so "nearly complete," or which are so easily "turned into complete sequences," that to call them "incomplete" seems little more than nitpicking. For example, the sequences $\{2, 3, 4, 5, 6, \ldots\}$ and $\{2, 2, 4, 6, 10, 16, \ldots\}$ are not complete, although every integer ≥ 2 is a sum of distinct terms of the first sequence, and although the sequence $\{1, 2, 2, 4, 6, 10, \ldots\}$, obtained by appending a 1 to the second sequence, is complete.

3. Weak Completeness

Definition 3: A sequence f is *weakly complete* [3] if a positive integer n exists such that every integer greater than n is a sum of distinct terms of f. Erdös & Graham [2] call such sequences *complete*.

A complete sequence is weakly complete. The sequence $f(n) = n + 1$, to give a trivial example, is weakly complete but not complete. The following theorem specifies a condition implying weak completeness.

Theorem 2: A sequence f is weakly complete if a positive integer n and a real number $s > 2$ exist such that:

i) If $x > n$, then there is a term of the sequence strictly between x and $(2 - 2/s)x$, and

ii) every integer between n and sn (inclusive) is a sum of distinct terms of the sequence.

Proof: By strong induction. Given an integer $w > sn$, we must show that w is a sum of distinct terms of the sequence. Let our induction hypothesis be that every integer inclusively between n and $w - 1$ is a sum of distinct terms of f.

There exists a term of the sequence, $f(t)$, strictly between $w/2$ and $(1 - 1/s)w$, by hypothesis i). Let

$$m = w - f(t).$$

Then $m < w/2$, and $m > w/s > n$. Since $n < m < w/2$, m is a sum of distinct terms of f; and, since $m < w/2 < f(t)$, none of these distinct terms equals $f(t)$. Since $w = m + f(t)$, w is a sum of distinct terms of f. By strong induction, the theorem is proved.

The two properties i) and ii) are not necessary for weak completeness. In particular, the function $f(n) = 2^{n-1}$ fails condition i) for all positive n and real $s > 2$. The sequence $f(n)$ is nevertheless complete. (I am obliged to the referee for this example.) The sequence $f(n) = n + 1$, on the other hand, satisfies i) and ii) for suitable s and n, and yet is incomplete. Thus, conditions i) and ii) are sufficient, but not necessary, for weak completeness, and are neither sufficient nor necessary for completeness.

The following examples of sequences which fail to be weakly complete show that this concept is not too broad.

Example 2: The Fibonacci sequence beginning with 2 and 2 is not weakly complete; neither is $\{2n\}$.

Proof: Let f be either of these sequences. Any term $f(n)$ is even, so any sum of distinct terms of f is even. No matter how large $n > 0$ is chosen, $2n + 1$ is greater than n and is not a sum of distinct terms of f.

If any two terms of the Fibonacci sequence $f_{1,1}$ are replaced by zeros, the resulting sequence is not weakly complete. A proof of this can be found in [3]. Thus, the Fibonacci sequence beginning with 2 and 3 is not weakly complete.

Definition 4: A sequence f is *finite* if a number n exists such that, for all natural numbers $m > n$, $f(n) = 0$. A sequence is *infinite* iff it is not finite. An infinite sequence f is *increasing* if, for any two natural numbers m and n such that $m > n$, $f(m) > f(n)$.

Definition 5: Let f be weakly complete. Then the greatest integer which is not a sum of distinct terms of f is called the *threshold* (of completeness) of f. Erdös & Graham [2] use the term "threshold" as well, but may not mean the same thing by it.

Theorem 3: The following conditions on a sequence f are equivalent.

a) Every infinite increasing sequence contains a term that is a sum of distinct terms of f.

b) Every infinite increasing sequence contains a subsequence each of whose terms is a sum of distinct terms of f.

c) f is weakly complete.

Proof: c) $\to$ b). Suppose c) holds. Let an infinite increasing sequence h be given. Then, if $h(m)$ is the least term of h greater than T, the threshold of f, then $g(n) = h(n + m)$ defines a subsequence of h each of whose terms is a sum of distinct terms of f.

b) $\to$ a). Obvious.

a) → c). If f were not weakly complete, then the sequence of numbers that are not sums of distinct terms of f would form an infinite increasing sequence containing no sum of distinct terms of f, so a) would not be true. So, if a) holds, then c) holds.

(I am obliged to the referee for suggestions which shortened this proof.)

4. Completability

The sequence {2, 2, 4, 6, 10, 16, 26, 42, 68, ...} is not weakly complete, even though it is "sufficiently rich" that the mere attachment of a 1 to this sequence renders it complete. This suggests the definition of a third, very general sort of completeness, called *completability*, such that a completable sequence becomes complete after a suitable finite sequence is prefixed to it, that is, attached to it at its beginning.

Definition 6: Suppose f is a sequence, and I is a finite sequence. If $I(n) = 0$ for all n, then define *the result of prefixing I to f* to be f. Otherwise, if m is a natural number such that $I(m)$ is nonzero and, if $n > m$, $I(n) = 0$, then define *the result of prefixing I to f* as the sequence h such that $h(n) = I(n)$ if $n \le m$, and $h(n) = f(n - m)$ if $n > m$.

The formal tools are now available with which to define completability:

Definition 7: A sequence f is *completable* if there exists a finite sequence I such that the result of prefixing I to f is complete.

Note that the completability of a sequence is not affected by the removal or prefixing of a finite number of terms from or to the sequence.

Theorem 4: A weakly complete sequence is completable.

Proof: Let f be weakly complete, and let T be its threshold (see Definition 5). Define the sequence I by letting

$$I(n) = n \text{ if } n \le T \quad \text{and} \quad I(n) = 0 \text{ if } n > T.$$

Then I is finite, and the result of prefixing I to f is complete.

The following two theorems derive sufficient conditions that a sequence be completable.

Theorem 5: Let f be a sequence. If a positive integer n and a real number v strictly between 1 and 2 exist such that, if $x > n$, there is a term of f strictly between x and vx, then f is completable.

Proof: Let $s = 2/(2 - v)$. Then $v = 2 - 2/s$, and $s > 2$. Define the sequence I to contain the integers between n and sn, inclusive, in numerical order, followed by zeros. The sequence I is finite. Let h be the result of prefixing I to f. If h is weakly complete, then it is completable by Theorem 4; hence, f is completable. Theorem 2 now applies: Our s is the s of that theorem.

The preceding theorem can be used to show that if f is a sequence, and if there exists a real number v strictly between 1 and 2 such that, for all sufficiently large n,

$$f(n) < f(n + 1) < vf(n),$$

then f is completable. If v is greater than 2 and the right-hand inequality is reversed, i.e., if

$$f(n + 1) > vf(n)$$

for all sufficiently large n, then f is not completable. This will be shown in Theorem 9.

Theorem 6: Let f be a sequence. Suppose there is a natural number $m > 1$ such that all but a finite number (possibly zero) of terms of f are divisible by m. Suppose, in addition, that the sequence I defined by

$$I(n) = f(n + s)/m,$$

where $f(s)$ is the last term of f that is not divisible by m, is complete. Then f is completable.

Proof: If there is a term of f not divisible by m, let r be the largest; if every term of f is divisible by m, let $r = 0$. Let

$$\begin{cases} h(n) = n & \text{if } n \le m, \\ h(n) = 0 & \text{otherwise,} \end{cases}$$

and let the sequence j be the result of prefixing the finite sequence h to f. We obtain that j is complete by a similar argument to that of the proof of Theorem 1.

Counterexample: The converse to Theorem 6 is false: there exists a completable sequence any two consecutive terms of which are relatively prime.

> Let f be the Fibonacci sequence whose first two terms are 2 and 3. Then f is completable because the result of prefixing the finite sequence h defined by $h(1) = 1$, $h(2) = 1$, and $h(n) = 0$ if $n > 2$, is complete.
>
> This shows that Theorem 6 does not characterize the completable sequences.

Theorem 6 proves that completable sequences can be obtained by multiplying every term of a complete sequence by a constant and prefixing some finite number, possibly zero, of terms. It is also true that, if every term of a weakly complete sequence is multiplied by a constant and a finite sequence is then prefixed, the result is completable (replace m by $m + T$, where T is the threshold of the weakly complete sequence, in the proof of Theorem 6). The concept of completability is certainly not restrictive. There is now the problem of proving that it is not too general—that the class of completable sequences does not coincide with the class of sequences. This will be done in the next four theorems.

Definition 8: Let f be a sequence. Then $P(f)$ is the set of all natural numbers that are sums of distinct terms of f. This notation is due to Erdös & Graham [2].

It follows from this definition that a sequence f is complete iff $P(f) = N$. Similarly, a sequence f is weakly complete iff $P(f)$ is cofinite (i.e., iff its complement in N is a finite set).

Theorem 7: A sequence f is completable iff there exists a positive integer c such that, if q is greater than c and is not in $P(f)$, then there exists a number m in $P(f)$ such that $0 < q - m \le c$.

Proof: ("Only if" part.) If f is weakly complete, then upon choosing c to be the threshold of f, the theorem follows trivially. Suppose f is completable but not weakly complete. If I is a finite sequence such that the result of prefixing I to f is complete, then let c be the maximum element of $P(I)$. Suppose q is greater than c and not in $P(f)$. Then q is the sum of distinct terms of I and distinct terms of f. Let the distinct terms in this sum from f, taken by themselves, have the sum m. The distinct terms in this sum from I are greater than zero, but cannot exceed c. However, q is the sum of m and these distinct terms of I, so $m < q \le m + c$. This implies $0 < q - m \le c$.

("If" part.) If I is the finite sequence consisting of c ones followed by zeros, then prefixing I to f we obtain a sequence g. Let q be a natural number. If $q \le c$, then q is the sum of q ones from I. If $q > c$, then either q is in $P(f)$ or is not. If q is not in $P(f)$, then there is m in $P(f)$ with $0 < q - m \le c$, and q is the sum of m in $P(f)$ and $q - m$ in $P(I)$. The terms whose sums are m and $q - m$, respectively, do not overlap because the terms of I precede the terms of f in the sequence. So every natural number is a sum of distinct terms of g. Thus, f is completable.

It follows from Theorem 7 that an infinite sequence f is complete iff the sequence h, defined by $h(n) = n^{\text{th}}$ term of $P(f)$ in order of magnitude, has the property that the difference between consecutive terms of h, $h(n + 1) - h(n)$, is a function of n that is bounded from above.

Theorem 7 is a necessary and sufficient condition that a sequence f be completable. The following theorem applies the contrapositive of the "only if" part of Theorem 7 to obtain a condition that a sequence not be completable.

Definition 9: A sequence f is *superincreasing* if the quantity

$$f(n) - \sum_{i=1}^{n-1} f(i)$$

is positive for all sufficiently large n. [Note that superincreasing sequences are increasing for n sufficiently large.]

Theorem 8: Let f be a superincreasing sequence. Suppose

$$f(n) - \sum_{i=1}^{n-1} f(i)$$

is unbounded from above. Then f is not completable.

Proof: Suppose the condition of Theorem 7 held. Then there would exist a number c such that, if n were greater than c and not in $P(f)$, there would exist m in $P(f)$ such that $0 < n - m \le c$. For all positive integers c, we will exhibit $t > c$ which is not in $P(f)$ and such that, if m is in $P(f)$ and is less than t, $c + m$ is also less than t, so that the sequence f cannot satisfy the necessary condition of Theorem 7.

Let $c > 0$ be given. By hypothesis, there are infinitely many n such that

$$f(n) - \sum_{i=1}^{n-1} f(i) > c + 1.$$

Choose any such n, and define $t = f(n) - 1$. Then:

a) t is not in $P(f)$.

For suppose t were in $P(f)$. Then t_i and r would exist such that

$$t = \sum_{i=1}^{r} f(t_i), \text{ for all } i,\ t_i < n,$$

since $t < f(n)$ and f is increasing beyond the n^{th} term. This implies that t, the sum of the terms $f(t_i)$ can be no greater than the sum of all terms up to the $(n-1)^{\text{th}}$, that is,

$$\sum_{i=1}^{n-1} f(i);$$

and so

$$t = \sum_{i=1}^{r} f(t_i) \le \sum_{i=1}^{n-1} f(i) < f(n) - c - 1 = t - c < t,$$

which is impossible.

b) $t > c$.

Since

$$f(n) - \sum_{i=1}^{n-1} f(i) - 1 > c,$$

and since $t = f(n) - 1$,

$$t > \sum_{i=1}^{n-1} f(i) + c \ge c.$$

c) If m is in $P(f)$ and $m < t$, then $m + c < t$.

Since $m < t$, $m < f(n)$. Since m is in $P(f)$, m is a sum of distinct terms of f, and since $m < f(n)$ this sum can be no greater than

$$\sum_{i=1}^{n-1} f(i).$$

So

$$c + 1 < f(n) - \sum_{i=1}^{n-1} f(i) \le f(n) - m;$$

hence,

$$c < t - \sum_{i=1}^{n-1} f(i) \le t - m \quad \text{and} \quad m + c < t.$$

Theorem 9: Let f be a sequence. Suppose there exists a real number $v > 2$ such that, for all sufficiently large n,

$$f(n+1) > vf(n).$$

Then f is not completable.

Proof: Let

$$h(n) = f(n) - \sum_{i=1}^{n-1} f(i) \quad (n \ge 2).$$

Suppose n is sufficiently large such that, for all $r \ge n$,

$$f(r+1) > vf(r).$$

Since,

$$f(r+1) > 2f(r),$$

$$f(r+1) - f(r) > f(r);$$

thus, subtracting

$$\sum_{i=1}^{r-1} f(i)$$

from both sides, we obtain

$$h(r+1) > h(r) \text{ for all } r \geq n.$$

We will show that the function h satisfies the condition of Theorem 8; that is, $h(r)$ is positive for sufficiently large r and unbounded from above.

Since

$$h(r+1) \geq h(r) + 1$$

for all $r \geq n$, it is true by induction that

$$h(r+m) \geq h(r) + m \text{ for all } m \geq 1 \text{ and } r \geq n.$$

Let $z = h(n)$. If $z \geq 0$, then $h(r)$ is positive for all $r > n$; so suppose $z < 0$. Then, if $m > -z$,

$$h(n+m) \geq h(n) + m > h(n) - z = 0,$$

so, if $r > n - z$, $h(r) > 0$. Thus, in any case, $h(r)$ is positive for all sufficiently large r.

Suppose $h(r)$ is bounded above by w, for all r. Again, let $z = h(n)$. Then $z < w$; let $m = w - z$. Then

$$h(n+m+1) \geq h(n) + m + 1 = z + m + 1 = w + 1,$$

a contradiction. So $h(r)$ is unbounded from above.

This theorem, and Theorem 5, relate completability to the rate of growth of a sequence. However, there are infinite sequences whose completability neither theorem can decide. For example, let f be the sequence defined by

$$f(n) = \begin{cases} 1, \text{ if } n = 1, \\ f(n-1)^2, \text{ if } n \text{ is even,} \\ f(n-1) + 1, \text{ if } n \geq 3 \text{ is odd.} \end{cases}$$

If n is sufficiently large, then

$$f(n+1) < 2f(n), \text{ if } n \text{ is even,}$$

$$f(n+1) > 2f(n), \text{ if } n \text{ is odd.}$$

f satisfies neither the hypothesis of Theorem 5 nor that of Theorem 9.

Theorem 9 yields infinite noncompletable sequences, for example, the sequence $f(n) = 3^n$.

Remark: Those results of the past two sections which relate the three definitions of completeness may be summarized as follows:

> Let J be the class of complete sequences, let K be the class of weakly complete sequences, let L be the class of completable sequences, and let M be the class of all infinite sequences (see Definition 4). Then $J \subset K \subset L \subset M$, and all containments are proper. The Remark after Definition 3, Theorem 4, and the Remark after Theorem 9 prove that $J \subseteq K$, $K \subseteq L$, $L \subset M$. The relations $J \subset K$ and $K \subset L$ are true because $f(n) = n + 1$ is in K but not in J, and because $f(n) = 2n$ is in L but not in K.

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3. Ross Honsberger. "A Second Look at the Fibonacci and Lucas Numbers." *Math. Gems III*, pp. 102-138. Mathematical Association of America Inc., 1985.

4. Torleiv, Kløve. "Sums of Distinct Elements from a Fixed Set." *Math. Comp.* (October 1975):1144-1149.

The following two articles are not utilized in my paper, but contain further information about complete sequences.

5. J. L. Brown, Jr. "A Characterization of 0-Sequences." *Math. Mag.* (September 1972):209-213.

A 0-sequence is a complete sequence which is rendered incomplete by the deletion of any term from the sequence. This article derives a necessary and sufficient condition that a nondecreasing sequence beginning with 1 be a 0-sequence.

6. J. L. Brown, Jr. "Integer Representations and Complete Sequences." *Math. Mag.* (January 1976):30-32.

This article derives a theorem which gives a necessary and sufficient condition that a nondecreasing sequence beginning with 1 be complete.

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*This paper was written by Eric Schissel while a senior at Roslyn High School, Roslyn, NY. The author is presently a student at Princeton University. This paper was written under the direction of Steven R. Conrad of Roslyn High School during the school year 1986-1987.