A BOX FILLING PROBLEM

Amitabha Tripathi
State University of New York at Buffalo
(Submitted January 1988)

1. Introduction

For an arbitrary but fixed integer $b > 1$, consider the set of ordered pairs $S_b = \{(i, a_i)\colon 0 \le i \le b - 1,\ a_i$ equals the number of occurrences of i in the sequence $a_0, a_1, \ldots, a_{b-1}\}$. A complete solution for S_b is given explicitly in terms of b. It is shown that there is a unique solution for each $b > 6$ and for $b = 5$, that there are two solutions for $b = 4$, and that there is none for $b = 2, 3$, or 6.

Let b be an arbitrary but fixed integer, $b > 1$. We wish to determine, whenever possible, the integers a_i $(0 \le i \le b - 1)$, where a_i denotes the number of occurrences of i in the lower row of boxes in the table below.

0	1	2	...	i	...	$b - 1$
a_0	a_1	a_2	...	a_i	...	a_{b-1}

This may be viewed as a problem in determining all possible sets whose members are functions that satisfy a special property. It is easy to see that the case $b = 2$ gives no solution; henceforth, we shall assume that $b \ge 3$. It is convenient to consider the cases $b > 6$ and $3 \le b \le 6$ separately.

2. The Case $b > 6$

It is clear from the definition of each a_i, that $a_0 \ne 0$. Thus, the set $T_b = \{a_i\colon a_i \ne 0\}$ is nonempty. In fact, $|T_b| = b - a_0$. Since a_i boxes are filled by i and since each box is necessarily filled by an integer at most $b - 1$, we have

$$\sum_{0 \le i \le b-1} a_i = b.$$

Define the set $T_{0,b} = T_b - \{a_0\}$. Clearly,

$$|T_{0,b}| = b - a_0 - 1 \quad \text{and} \quad \sum a_i = b - a_0.$$

Since each member of $T_{0,b}$ is at least 1, it follows that $T_{0,b}$ consists of $(b - a_0 - 2)$ 1's and one 2. (*)

If $a_0 = 1$, $T_{0,b}$ would consist of $(b - 3)$ 1's and one 2, and T_b would consist of $(b - 2)$ 1's and one 2. This is impossible since the boxes are being filled by 0, 1, and 2, while $a_1 = b - 2 > 4$.

If $a_0 = 2$, $T_{0,b}$ would consist of $(b - 4)$ 1's and one 2, and T_b would consist of $(b - 4)$ 1's and two 2's. This, too, is impossible since the boxes are being occupied by 0, 1, and 2, while $a_1 = b - 4 > 2$.

Thus, $a_0 \geq 3$ and $a_A = 1$ where $A = a_0$. Hence,

$$T_b = \{a_0,\ a_1 = b - a_0 - 2,\ a_2 = 1,\ a_A = 1\}.$$

But $|T_b| = b - a_0 = 4$ implies that $a_0 = b - 4$ and the unique solution in this case is given in the table below.

0	1	2	3	...	$b - 5$	$b - 4$	$b - 3$	$b - 2$	$b - 1$
$b - 4$	2	1	0	...	0	1	0	0	0

3. The Case $b \leq 6$

By repeating the argument in the case $b > 6$ until (*), if $a_0 \neq 1$ or 2, we would have $|T_b| = b - a_0 = 4$ and so $b = a_0 + 4 \geq 7$. Hence, $a_0 = 1$ or 2.

If $a_0 = 1$, T_b would consist of $(b - 2)$ 1's and one 2. Since all the boxes are being occupied by 0, 1, and 2, we must have $b - 2 \leq 2$. If $b = 3$, we have $a_0 = a_1 = a_2 = 1$, which does not give a solution. If $b = 4$, we have $a_0 = 1$, $a_1 = 2$, $a_2 = 1$, which does give a solution.

If $a_0 = 2$, T_b would consist of $(b - 4)$ 1's and two 2's. Since all of the boxes are filled by 0, 1, and 2, we must have $b - 4 \leq 2$. If $b = 4$, we have $a_0 = 2$, $a_1 = 0$, $a_2 = 2$, which gives a solution. If $b = 5$, we have $a_0 = a_1 = a_2 = 2$, which does not give a solution.

We thus have two solutions if $b = 4$, one solution if $b = 5$, and no solution if $b = 2$, 3, or 6, and these are listed in the tables below.

0	1	2	3
1	2	1	0
2	0	2	0

$b = 4$

0	1	2	3	4
2	1	2	0	0

$b = 5$

Acknowledgment

The author wishes to thank Michael Esser for having suggested the problem for the case $b = 10$.

Reference

1. H. J. Ryser. *Combinatorial Mathematics*. The Carus Mathematical Monographs #14. New York: The Mathematical Association of America, 1965.

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