ON K^{th}-ORDER COLORED CONVOLUTION TREES AND A GENERALIZED ZECKENDORF INTEGER REPRESENTATION THEOREM

J. C. Turner
University of Waikato, Hamilton, New Zealand

A. G. Shannon
University of Technology, Sydney 2007, Australia
(Submitted October 1987)

1. Introduction

In Turner [6], a sequence of trees was defined such that the n^{th} tree T_n had F_n leaf-nodes, where F_n is the n^{th} element of the Fibonacci sequence. It was shown how to construct the trees so that the nodes were weighted with integers from a general sequence $\{C_n\}$ using a sequential method described in Section 2.

This produced a sequence of Fibonacci convolution trees $\{T_n\}$, so called because the sum of the weights assigned to the nodes of T_n was equal to the nth term of the convolution product of $\{F_n\}$ and $\{C_n\}$. That is, the Ω meaning the sum of weights:

$$\Omega(T_n) = (F * C)_n = \sum_{i=1}^{n} F_i C_{n-i+1}.$$

This result is illustrated in Section 3.

With this construction, a graphical proof of a dual of Zeckendorf's theorem was given, namely that every positive integer can be represented as the sum of distinct Fibonacci numbers, with no gap greater than one in any representation, and that such a representation is unique [2].

To develop this procedure further, we give a construction for k^{th}-order colored trees, and for colors consider generalized Fibonacci numbers of order 2 and greater. To this end, we define the recurring sequence

$$\{W_n\} = \{W_n(a, b; p, q)\}$$

as in Horadam [4] by the homogeneous linear recurrence relation

$$W_n = pW_{n-1} - qW_{n-2}, \ n > 2,$$

with initial conditions $W_1 = a$, $W_2 = b$. The ordinary Fibonacci numbers are then

$$\{F_n\} \equiv \{W_n(1, 1; 1, -1)\}.$$

2. Construction of K^{th}-Order Colored Trees

Given a sequence of colors, $C = \{C_1, C_2, C_3, \ldots\}$, we construct k^{th}-order colored trees, T_n, as follows:

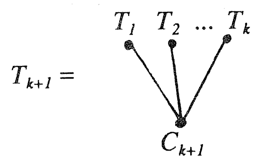
$T_1 = C_1^{\bullet}$

$T_n = T_{n-1}^{\bullet}\!\!-\!\!\!-\!\!\!-^{\bullet} C_n$, with $C_1^{\bullet}$ being the root node in each case,

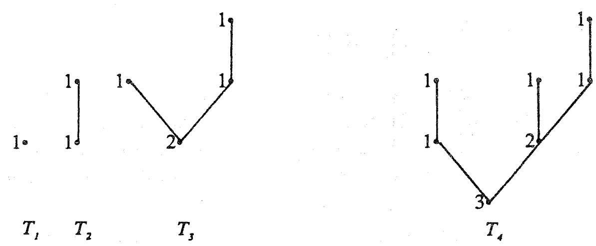
$n = 2, 3, \ldots, k;$

$$T_{k+m} = C_{k+m} \overset{k-1}{\underset{i=0}{\mathbf{V}}} T_{m+i}, \quad m = 1, 2, \ldots$$

in the "drip-feed" construction, in which the k^{th}-order fork operation V is to mount trees T_m, T_{m+1}, ..., T_{m+k-1} on separate branches of a new tree with root node at C_{m+k} for $m \geq 1$. Thus, when $m = 1$, we get:



For example, when $k = 2$ and $C = \{F_n\}$, the sequence of Fibonacci convolution trees is

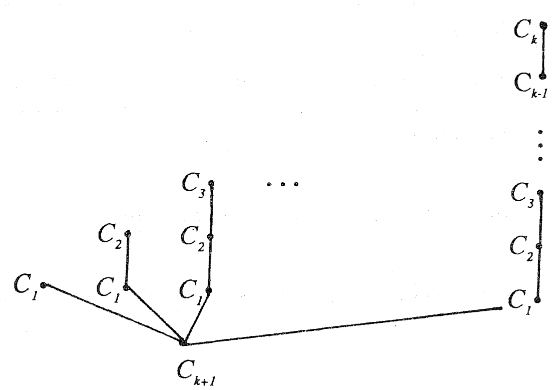


where T_1, T_2, are the initial trees, and subsequent ones are:

$$T_3 = F_3 \overset{1}{\underset{i=0}{\mathbf{V}}} T_{i+1} = F_3 V(T_1, T_2),$$

$$T_4 = F_4 V(T_2, T_3), \text{ and so on (see Turner [8]).}$$

The tree T_{k+1} for the general case is:



In Section 4, we take the colors from the k^{th}-order Fibonacci sequence (cf. Shannon [5]) given by

$$C_{n+k} = \sum_{i=0}^{k-1} C_{n+i}, \quad n \geq 1, \text{ and with initial elements } C_1, C_2, \ldots, C_k.$$

The colors on the leaves taken from all the trees in the sequence, from left to right, form the Fibonacci word pattern (Turner [8]):

$$f(C_1,\ C_2,\ \ldots,\ C_k) = C_1,\ C_2,\ \ldots,\ C_k,\ C_1C_2 \ldots C_k,\ C_2 \ldots C_kC_1 \ldots C_k,\ \ldots$$

There is a remarkable relationship between the leaf-word pattern and the tree shade sets, which we shall discuss in Section 4.

Examples of Fibonacci word patterns, showing how they are formed by r^{th}-order word-juxtaposition recurrences are,

$$r = 2:\quad f(a,\ b) = a,\ b,\ ab,\ bab,\ abbab,\ \ldots$$
$$r = 3:\quad f(a,\ b,\ c) = a,\ b,\ c,\ abc,\ bcabc,\ cabcbcabc,\ \ldots$$
$$r = 4:\quad f(a,\ b,\ c,\ d) = a,\ b,\ c,\ d,\ abcd,\ bcdabcd,\ \ldots\ .$$

It is of interest to note that

$$f(a,\ b) = \{W_n(a,\ b;\ 1,\ -10 \uparrow F_{n-1})\}$$

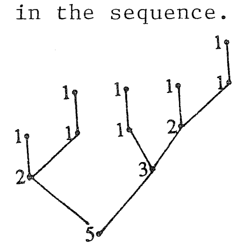
where $10 \uparrow m$ represents 10^m for notational convenience, $\{W_n\}$ is Horadam's generalized Fibonacci sequence, and F_n are the ordinary Fibonacci numbers. Thus,

$$W_1 = a$$
$$W_2 = b,$$
$$\begin{aligned} W_3 &= W_2 + 10^1W_1 \\ &= b + 10a \\ &= ab, \text{ in the above notation, and so on.} \end{aligned}$$

3. Number Properties of the Trees

(i) Node weight totals

As stated in the Introduction, when $k = 2$ the sum of all node weights of T_n is equal to the convolution term $(F * C)_n$. We illustrate this for the case $C = \{F_n\}$ and with the fifth tree in the sequence.

$T_5 =$ 

From observation,

$$\Omega(T_5) = 1 + (1 + 1 + 1 + 1) + (1 + 1 + 1 + 2) + (2 + 3) + 5 = 20.$$

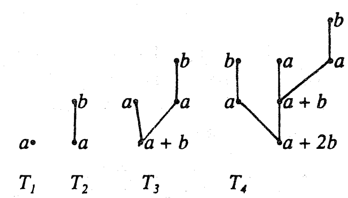
Using the formula, we get

$$\begin{aligned} \Omega(T_5) &= F_1F_5 + F_2F_4 + F_3F_3 + F_4F_2 + F_5F_1 \\ &= 2F_1F_5 + 2F_2F_4 + F_3^2 \\ &= 10 + 6 + 4 = 20. \end{aligned}$$

(ii) Number of nodes with colors $C_1,\ C_2,\ \ldots,\ C_k$ at different nodes

Now consider the first four trees associated with the color sequence

$$C = F(a,\ b) = a,\ b,\ a + b,\ a + 2b,\ \ldots$$



Note that we use $f(a, b)$ to denote the word pattern and $F(a, b)$ for the color sequence.

Let (n_a, n_b) represent the number of a's and the number of b's at any level in the tree; we get the following table for this pair.

Tree T_n	m = (level + 1)	1	2	3	4	5	6
T_1		(1, 0)					
T_2		(1, 0)	(0, 1)				
T_3		(1, 1)	(2, 0)	(0, 1)			
T_4		(1, 2)	(2, 1)	(2, 1)	(0, 1)		
T_5		(2, 3)	(2, 3)	(4, 1)	(2, 2)	(0, 1)	
T_6		(3, 5)	(3, 5)	(4, 4)	(6, 2)	(2, 3)	(0, 1)

If we represent the element in the n^{th} row and m^{th} column of this array by x_{nm}, then x_{nm} satisfies the partial recurrence relation

$$x_{nm} = x_{n-1,m-1} + x_{n-2,m-1}, \quad 1 < m < n, \; n > 2,$$

with boundary conditions

$$x_{11} = x_{21} = (1, 0);$$

$$x_{22} = (0, 1);$$

$$x_{n1} = (F_{n-2}, F_{n-1}), \; n > 2;$$

$$x_{nm} = (0, 0), \; m > n.$$

The proof follows from the construction of the trees and the fact that the root color for T_n, after $n = 2$, is the n^{th} term of $f(a, b)$, which is $aF_{n-2} + bF_{n-1}$ (see Horadam [3]).

4. Shades and Leaf Patterns

Consider the set of all leaf-to-root paths in a given convolution tree. Each leaf node determines a unique path, say P_i. We can label the paths P_1, P_2, ..., P_g according to their position (taken from left to right) on the tree diagram. If we add up the node weights on path P_i and denote this path weight by W_i, we obtain a sequence $\{W_1, W_2, \ldots, W_g\}$, called the shade of the tree (Turner [6]). This is denoted by $Z(T_n)$ and is described in more detail in Section 5.

Recall from Section 2 that the colors on the leaves of the trees in the sequence form a Fibonacci word pattern. For example, the pattern $f(1, 4)$ of leaf nodes for $k = 2$ can be seen in Figure 1.

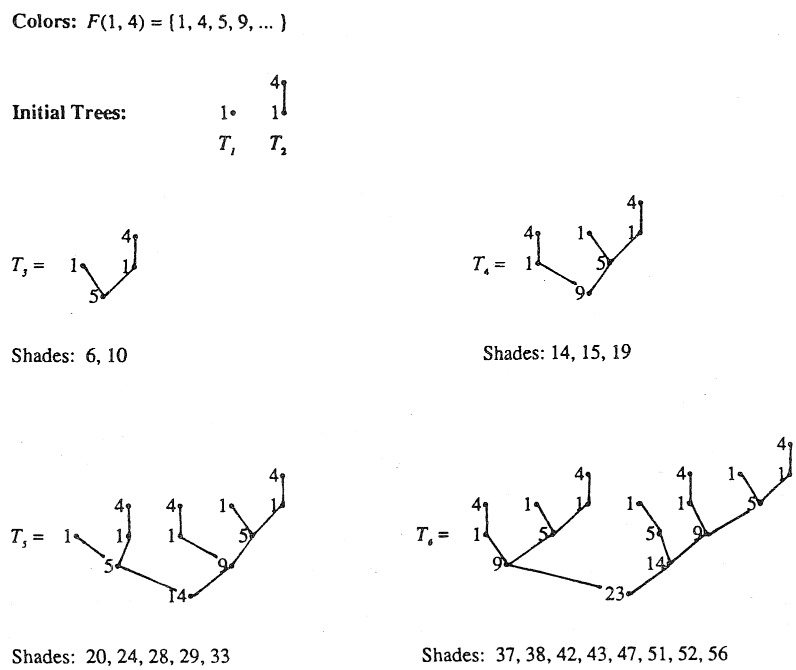


FIGURE 1

The shades can be generated by the ϕ function of Atanassov [1] defined by $\phi: \mathbb{N} \to \mathbb{N}$ such that $\phi(0) = 0$ and

$$\phi(n) \equiv \phi(a_1 a_2 \ldots a_k) = \sum_{i=1}^{k} a_i,$$

where

$$n = \sum_{i=1}^{k} 10^{k-i} a_i.$$

The shades and leaf numbers of the first four trees for $f(1, 4)$ are as follows:

Tree	T_1	T_2	T_3	T_4	...
Leaf numbers	1	4	14	414	...
Shade	1	5	6, 10	14, 15, 19	
	$\phi(1) = 1$	$\phi(14) = 5$	$\phi(51) = 6$	$\phi(10, 4) = 14$	
			$\phi(64) = 10$	$\phi(14, 1) = 15$	
				$\phi(15, 4) = 19$	

Note that $\phi(n)$ just accumulates the elements of the total leaf number pattern

$$f(1, 4) = 1, 4, 14, 414, 14414, \ldots,$$

from the left. Thus, the shade set for the tree sequence is

$$\begin{array}{r} 1 \\ 1 + 4 = 5 \\ 1 + 4 + 1 = 6 \\ 1 + 4 + 1 + 4 = 10\ldots \end{array}$$

In the general second-order case,

$$f(a, b) = a, b, ab, bab, \ldots,$$

and the shade sequence $r_i a + s_i b$ is, in turn,

$$1a,\quad \begin{matrix}1a,\\+\\1b\end{matrix}\quad \begin{matrix}2a,\\+\\1b\end{matrix}\quad \begin{matrix}2a,\\+\\2b\end{matrix}\quad \begin{matrix}2a,\\+\\3b\end{matrix}\quad \begin{matrix}3a,\\+\\3b\end{matrix}\quad \begin{matrix}3a,\\+\\4b.\end{matrix}$$

An example can be seen in Figure 1 for $f(1, 4)$.

Various results can be developed from the coefficients of a and b in this sequence. For example, if we write them as two-component vectors, we get

$$\left\{\binom{r_i}{s_i}\right\} = \left\{\binom{1}{0}, \binom{1}{1}, \binom{2}{1}, \binom{2}{2}, \binom{2}{3}, \binom{3}{3}, \binom{3}{4}, \binom{4}{4}, \binom{4}{5}, \ldots\right\}$$

then the first differences are:

$$\begin{aligned}\{\Delta_i\} &= \left\{\binom{r_{i+1}}{s_{i+1}} - \binom{r_i}{s_i}\right\}\\ &= \left\{\binom{0}{1}, \binom{1}{0}, \binom{0}{1}, \binom{0}{1}, \binom{1}{0}, \binom{0}{1}, \binom{1}{0}, \binom{0}{1}, \binom{0}{1}, \ldots\right\}\\ &= f\left(\binom{0}{1}, \binom{1}{0}, \binom{0}{1}\right).\end{aligned}$$

Note that the elements of $\{\Delta_i\}$ determine the Wythoff pairs, much studied in the Fibonacci literature [7]. To see this, consider the positions of the 1's in the upper elements of $\{\Delta_i\}$, and likewise in the lower elements: the upper 1's indicate the sequence $\{[n\alpha^2]\}$, and the lower 1's the sequence $\{[n\alpha]\}$.

It is now clear that for the leaf number pattern

$$f(1, 1) = 1, 11, 111, 11111, \ldots$$

the shade is

$$1, 23, 456, 7891011, \ldots,$$

as Figure 2 so graphically illustrates: for we merely have to accumulate the sequence of 1's, from the left, to get the shade, which is the sequence at the base of the straight lines from the trees to the horizontal axis.

Thus, each natural number n corresponds to a leaf-to-root path; and the path's color-sum provides a representation of n as a sum of distinct Fibonacci numbers:

$$n = \sum e_i F_i, \qquad e_i \in \{0, 1\}.$$

Furthermore, $e_i + e_{i+1} > 0$ for each i, which means that there is never a gap greater than one among the Fibonacci numbers constituting any representation, which is evident from the "drip-feed" tree-coloring procedure. Deleting the 1 from each leaf node, in each representation, one obtains integer representations with the same properties but in terms of distinct members of the sequence $\{u_n\} = \{F_{n+1}\}$. This integer-representation result has come to be known as Zeckendorf's dual theorem [2].

We now present two general results about the leaf patterns and shade sets.

Theorem 1: For the k^{th}-order tree sequence defined in Section 2, the colors on the leaves, from left to right, form the Fibonacci word-pattern

$$f(C_1, C_2, \ldots, C_k).$$

Proof: The colors on the leaves, from left to right, are initially by construction $C_1, C_2, \ldots, C_k$ in turn, and then for T_{k+1} they are $C_2 \ldots C_k C_1$, and so on, as in the recurrence that produces the k^{th}-order Fibonacci word-pattern.

Corollary: It follows that if C is the k^{th}-order Fibonacci sequence with initial elements $C_1, C_2, \ldots, C_k$ as in Section 2, then

$$\sum_{T_n} (\text{leaf colors}) = (\text{root color } C_n),\ n > k.$$

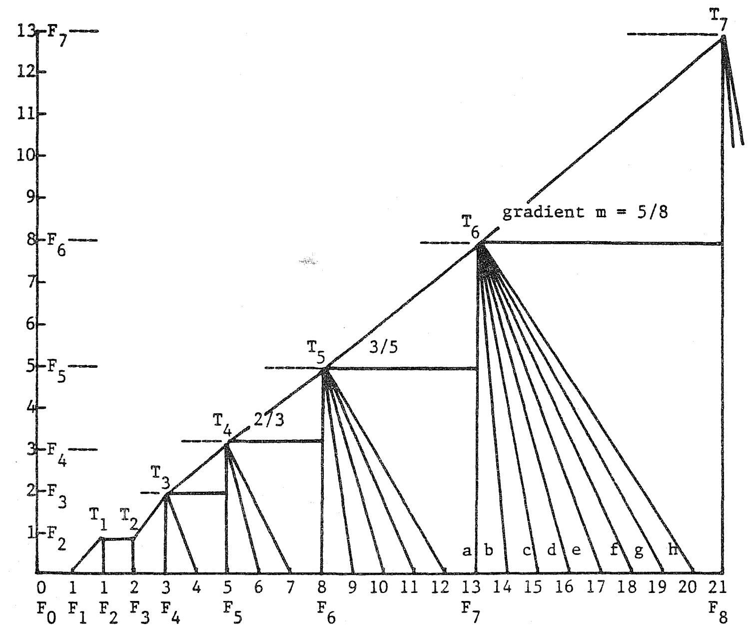


FIGURE 2(a)

Weights of leaf-to-root paths versus max. node weight

a:1 1 3 8 = 13
b:1 2 3 8 = 14
c:1 1 2 3 8 = 15
d:1 2 5 8 = 16
e:1 1 2 5 8 = 17
f:1 1 3 5 8 = 18
g:1 2 3 5 8 = 19
h:1 1 2 3 5 8 = 20

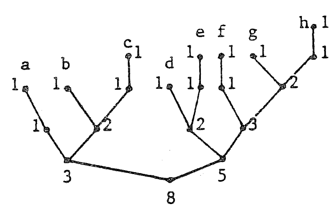


FIGURE 2(b)

Leaf-to-root paths for T_6

Theorem 2: The shade set of T_n (from the sequence of Theorem 1 and its corollary) is given by adding leaf-colors from the left, that is, by computing the partial sums of the leaf-pattern. Thus, if the leaf nodes of T_n have the color pattern $L_1L_2 \ldots L_r$ with each $L_i \in \{C_1, C_2, \ldots, C_k\}$, then the shade

$$Z(T_n) = \{S_1 + R_{n-1}, S_2 + R_{n-1}, \ldots, S_r + R_{n-1}\} \text{ for } n > k,$$

where L_m is the m^{th} partial sum of the leaf color pattern (left to right) of T_n, and

$$R_{n-1} = \sum_{i=1}^{n-1} C_i$$

is the sum of the root colors for the previous $n - 1$ trees.

Proof: An inductive proof is easily established.

Corollary: If $C_i = 1$ for $i = 1, \ldots, k$,

$$\lim_{n\to\infty} \bigcup_n Z(T_n) = Z^+, \text{ for any } k \geq 2.$$

This corollary provides the integer representations which are the subject of the next section.

We can also represent the shades in terms of $\{W_n\}$, as defined in Section 1. For $k = 2$ and $f(1, 1)$, we can define the sequence $\{S_{nm}\}$ by

$$S_{nm} \equiv 10W_n + 1 \pmod{10 \uparrow m},\ 1 \leq m \leq n.$$

Then, for example, for $\{W_n\} \equiv \{W_n(1, 4; 1, -10 \uparrow F_n)\}$, we have

n	1	2	3	4	5
W_n	1	4	14	414	14414

and

m	1	2	3	4	5
S_{5m}	1	41	141	4141	44141
$\phi(S_{5m})$	1	5	6	10	14

which is the shade sequence we found in Section 3.

5. Integer Representation Theorem

A family of integer representations using the k^{th}-order Fibonacci sequence with 1's for the first k elements is given by the following theorem.

Theorem 3: Any integer $n \in Z^+$ has a representation of the form

$$n = \sum e_i C_i, \quad e_i \in \{0, 1\}.$$

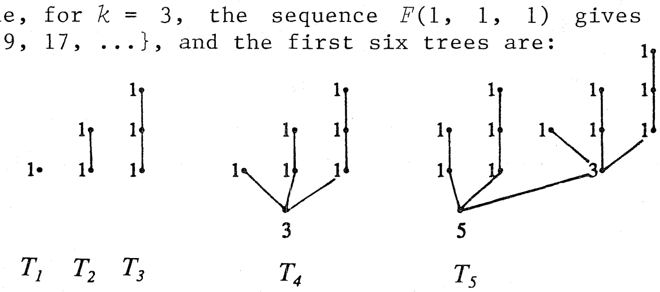
where the C_i are distinct elements of the k^{th}-order Fibonacci sequence $F(1, 1, \ldots, 1)$, and

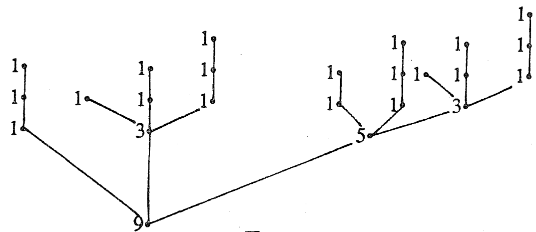
$$\sum_{j=0}^{k-1} e_{i+j} > 0 \text{ for all } i,\ k \geq 1.$$

Proof: The proof follows immediately from Theorem 2 and its corollary and the manner of construction of the trees. (The Zeckendorf dual occurs when $k = 2$.)

Corollary: We can use the initial 1's in each representation in a manner which provides representations for all integers in terms of distinct elements of the sequence whose first elements are 1, 2, 3, ..., k, and whose subsequent elements are the corresponding 1's of $F(1, 1, 1, \ldots, 1)$.

As an example, for $k = 3$, the sequence $F(1, 1, 1)$ gives the color set $\{1, 1, 1, 3, 5, 9, 17, \ldots\}$, and the first six trees are:





T_6

The following table shows the shades and the corresponding integer representations for integers N = 1, ..., 15 when the initial 1's are replaced by a 2 when (1, 1) occurs and by 1, 2 when (1, 1, 1) occurs in a representation.

T_n	1	2	3	4	5	6
$Z(T_n)$	1	2	3	4 5 6	7 8 9 10 11	12 13 14 15 ...
Integers in Representation	1	2	3	1 2 1 3 3 2 3	2 3 1 2 1 5 5 3 3 2 5 5 3 5	1 1 2 1 2 3 3 2 9 9 9 3 9
Maximum gap	-	-	-	1 0 0	1 0 1 0 0	2 1 1 1

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