HURWITZ'S THEOREM AND THE CONTINUED FRACTION WITH CONSTANT TERMS

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Introduction

We are concerned with finding the convergents

$$C_j(\alpha) = \frac{p_j}{q_j},$$

in lowest terms, to the positive real number α which satisfy the inequality relating to Hurwitz's theorem,

$$|\alpha - C_j(\alpha)| < \frac{\beta}{\sqrt{5}q_j^2}, \quad 0 < \beta < 1, \tag{1}$$

where α has a simple continued fraction expansion $\{i; i, i, \ldots\}$ and i is a positive integer.

Van Ravenstein, Winley, & Tognetti [5] have solved this problem for the case where $i = 1$, which means α is the Golden Mean, and extended that result in [6] to the case where α is a Noble Number that is a number equivalent to the Golden Mean.

The Markov constant for α, $M(\alpha)$, is defined at the upper limit on $\sqrt{5}/\beta$ such that (1) has infinitely many solutions p_j, q_j (see Le Veque [4]). Thus, in order to determine $M(\alpha)$, we require the lower limit on values of β such that there are infinitely many solutions.

Using the notation of [6] and the well-known facts concerning simple continued fractions (see Chrystal [2], Khintchine [3]), we have:

(i) If $\alpha = \{i; i, i, \ldots\}$ where i is an integer and $i \geq 1$, then

$$\alpha = \frac{i + \sqrt{i^2 + 4}}{2},$$

which is the positive root of the equation $x^2 - ix - 1 = 0$;

(ii) $$p_j = \frac{\left(\alpha^{j+2} - \left(-\frac{1}{\alpha}\right)^{j+2}\right)}{\left(\alpha + \frac{1}{\alpha}\right)}, \quad q_j = \frac{\left(\alpha^{j+1} - \left(-\frac{1}{\alpha}\right)^{j+1}\right)}{\left(\alpha + \frac{1}{\alpha}\right)} = p_{j-1} \tag{2}$$

where $j = 0, 1, 2, \ldots$.

Hence, $$C_j(\alpha) = \frac{p_j}{q_j} = \frac{\left(\alpha^{j+2} - \left(-\frac{1}{\alpha}\right)^{j+2}\right)}{\left(\alpha^{j+1} - \left(-\frac{1}{\alpha}\right)^{j+1}\right)}.$$

The numbers p_j have been studied extensively by Bong [1] where their relationship with Fibonacci and Pell numbers is described in detail.

Solutions to (1)

Case 1. If j is odd ($j = 2k + 1$, $k = 0, 1, 2, \ldots$), then (1) becomes

$$q_j(p_j - \alpha q_j) < \frac{\beta}{\sqrt{5}}$$

which, using (2)(ii), finally reduces to

$$\left(\frac{1}{\alpha^4}\right)^k > \alpha^4\left(1 - \frac{\beta}{\sqrt{5}}\left(\alpha + \frac{1}{\alpha}\right)\right). \tag{3}$$

From (3), we see that;

(i) there are no solutions for k if

$$0 < \beta \le \frac{\sqrt{5}(\alpha^2 - 1)}{\alpha^3}; \tag{4}$$

(ii) there is a nonzero finite number of solutions for k if

$$0 < \alpha^4\left(1 - \frac{\beta}{\sqrt{5}}\left(\alpha + \frac{1}{\alpha}\right)\right) < 1,$$

which simplifies to

$$0 < \frac{\sqrt{5}(\alpha^2 - 1)}{\alpha^3} < \beta < \frac{\sqrt{5}}{\left(\alpha + \frac{1}{\alpha}\right)} \le 1. \tag{5}$$

We note that equality holds on the right in (5) only when α is the Golden Mean.

(iii) All nonnegative integers are solutions for k if

$$\frac{\sqrt{5}}{\left(\alpha + \frac{1}{\alpha}\right)} \le \beta < 1. \tag{6}$$

Case 2. If j is even ($j = 2k$, $k = 0, 1, 2, \ldots$), then (1) becomes

$$q_j(\alpha q_j - p_j) < \frac{\beta}{\sqrt{5}}$$

and again using (2)(ii), this reduces to

$$\left(\frac{1}{\alpha^4}\right)^k < \alpha^2\left(\frac{\beta}{\sqrt{5}}\left(\alpha + \frac{1}{\alpha}\right) - 1\right). \tag{7}$$

From (7), we see that:

(i) there are no solutions for k if

$$0 < \beta \le \frac{\sqrt{5}}{\left(\alpha + \frac{1}{\alpha}\right)}; \tag{8}$$

(ii) there is a nonzero finite number of nonsolutions for k if

$$0 < \alpha^2\left(\frac{\beta}{\sqrt{5}}\left(\alpha + \frac{1}{\alpha}\right) - 1\right) < 1,$$

which simplifies to

$$0 < \frac{\sqrt{5}}{\left(\alpha + \frac{1}{\alpha}\right)} < \beta < \frac{\sqrt{5}}{\alpha}; \tag{9}$$

(iii) all nonnegative integers are solutions for k if

$$\frac{\sqrt{5}}{\alpha} \le \beta < 1. \tag{10}$$

In the particular case $i = 1$, α is the Golden Mean, $\alpha + (1/\alpha) = \sqrt{5}$, and there will be no convergents $C_j(\alpha)$ that satisfy (1) when j is even. However, if $i \ge 2$, then $(\sqrt{5}/\alpha) < 1$ and there are convergents that satisfy (1) when j is even.

Summary

Define

$$\beta_L = \frac{\sqrt{5}(\alpha^2 - 1)}{\alpha^3}, \quad \beta_M = \frac{\sqrt{5}}{\left(\alpha + \frac{1}{\alpha}\right)}, \quad \beta_U = \frac{\sqrt{5}}{\alpha}$$

Using (4)-(10), we see that:

(i) If $i \ge 2$, then $\beta_L < \beta_M < \beta_U < 1$ and there are no convergents that satisfy (1) when $0 < \beta \le \beta_L$.

If $\beta_L < \beta < \beta_M$, there are a finite number of convergents $C_j(\alpha)$ that satisfy (1) with $j = 1, 3, 5, \ldots, 2[R] + 1$ and

$$R = \frac{\ln\left\{\alpha^4\left(1 - \frac{\beta}{\sqrt{5}}\left(\alpha + \frac{1}{\alpha}\right)\right)\right\}}{\ln\left(\frac{1}{\alpha^4}\right)}. \tag{11}$$

If $\beta = \beta_M$, there are an infinite number of convergents that satisfy (1) given by all $C_j(\alpha)$ where j is odd.

If $\beta_M < \beta < \beta_U$, there are an infinite number of solutions to (1). These are given by all $C_j(\alpha)$ for j odd and all but a finite number of $C_j(\alpha)$ when $j = 0, 2, 4, \ldots, 2[S]$ where

$$S = \frac{\ln\left\{\alpha^2\left(\frac{\beta}{\sqrt{5}}\left(\alpha + \frac{1}{\alpha}\right)\right) - 1\right\}}{\ln\left(\frac{1}{\alpha^4}\right)}. \tag{12}$$

If $\beta_U \le \beta < 1$, there are an infinite number of solutions to (1) given by $C_j(\alpha)$ for $j = 0, 1, 2, \ldots$.

(ii) If $i = 1$, then $\beta_L < \beta_M = 1 < \beta_U$ and there are no convergents that satisfy (1) unless $\beta_L < \beta < 1$. In this case, the only convergents that are solutions to (1) are given by

$$C_j(\alpha) = \frac{F_{j+1}}{F_j}, \; j = 1, 3, 5, \ldots, 2[R] + 1,$$

where

$$R = \ln\frac{(1 - \beta)(7 + 3\sqrt{5})}{2} \Big/ \ln\frac{(7 - 3\sqrt{5})}{2} \text{ as specified in [5].} \tag{13}$$

(iii) The lower limit on numbers β such that (1) has infinitely many solutions is given by

$$\beta_M = \frac{\sqrt{5}}{\left(\alpha + \frac{1}{\alpha}\right)},$$

and in this case the Markov constant for α is given by

$$M(\alpha) = \frac{\sqrt{5}}{\beta_M} = \alpha + \frac{1}{\alpha} = \sqrt{i^2 + 4}. \tag{14}$$

Examples

1. If $i = 2$, then $\alpha = 1 + \sqrt{2} = \{2; 2, 2, \ldots\}$, $\beta_L \simeq 0.77$, $\beta_M \simeq 0.79$, $\beta_U \simeq 0.93$. Hence, we see that for:

   (i) $\beta \in (0, 0.77]$, there are no convergents satisfying (1);

   (ii) $\beta \in (0.77, 0.79)$, there are a finite number of convergents satisfying (1) and these are specified by (11);

   (iii) $\beta = 0.79$, there are an infinite number of convergents satisfying (1) given by all $C_j(\alpha)$ where $j = 1, 3, 5, \ldots$;

   (iv) $\beta \in (0.79, 0.93)$, all the convergents $C_j(\alpha)$ satisfy (1) for j odd, whereas all but those specified by (12) satisfy (1) for j even;

   (v) $\beta \in (0.93, 1)$, all convergents satisfy (1).

   In particular, it is seen from (14) that $M(1 + \sqrt{2}) = 2\sqrt{2}$.

2. If $\alpha = \{1; 1, 1, 1, \ldots\} = \dfrac{1 + \sqrt{5}}{2}$, then $\beta_L \simeq 0.85$, $\beta_M = 1$, $\beta_U \simeq 1.38$.

   Consequently, if $\beta \in (0, 0.85]$, there are no convergents that satisfy (1), whereas, if $\beta \in (0.85, 1)$, there are a finite number of solutions to (1) specified by (13). If $\beta = 1$, there are an infinite number of solutions given by all $C_j(\alpha)$ where j is odd and we see from (14) that

$$M\left(\frac{1 + \sqrt{5}}{2}\right) = \sqrt{5}.$$

References

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