ON FRIENDLY-PAIRS OF ARITHMETIC FUNCTIONS

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1. In [1] it is shown that there exists a "friendly-pair" of multiplicative functions $\{f, g\}$ such that

(1.1) $f(n^\alpha) = g(n),\ g(n^\alpha) = f(n),\ f(n)g(n) = 1$

for a fixed integer $\alpha \geq 2$. It is clear that f and g must satisfy the functional relation,

(1.2) $F(n^{\alpha^2}) = F(n)$ for all natural numbers n.

Hence, it is natural to examine whether pairs of functions $\{f, g\}$, not necessarily multiplicative, exist so that

(1.3) $f(n^\alpha) = g(n),\ g(n^\beta) = f(n)$

for a given pair $\alpha, \beta \geq 1$. Relation (1.3) implies that f and g must both satisfy the following functional equation where $r = \alpha \cdot \beta$.

(1.4) $F(n^r) = F(n)\ \forall\ n \in \mathbb{N}$ (the set of all natural numbers).

Conversely, if F satisfying (1.4) for some r exists, then for any factorization of r as $\alpha \cdot \beta$ we could define

(1.5) $f(n) = F(n),\ g(n) = F(n^\alpha)$ so that $g(n^\beta) = f(n)$

and so f and g satisfy (1.3). N.B. If r is prime, then both f and g are the same as F defined by (1.4).

Thus, it suffices to look for arithmetic functions F that satisfy what may be called the "power-periodicity" expressed in (1.4).

2. A complete characterization of such a power-periodic function F is more straightforward than when F is required to be multiplicative: Given a natural number $r > 1$, define $F(m)$ arbitrarily for every m that is not an r^{th} power of a natural number. Every natural number n that is an r^{th} power is uniquely expressible as

(2.1) $n = m^{r^i}$, m a non-r^{th} power and i a natural number.

So $F(n)$ with power-period r is easily characterized by its values at non-r^{th} powers.

3. Suppose F is required to be multiplicative. Then (1.4) implies:

(3.1) $\prod_{p|n} F(p^{ra}) = \prod_{p|n} F(p^a)$ where $n = \prod p^a$

in the standard form of unique factorization into powers of primes. Writing $F(p^a)$ as $G_p(a)$ and considering G as an arithmetic function of a, we are led to the following property of G that would suffice to ensure the power-periodicity of F.

Define a "multiplicatory-periodic" arithmetic function with period r by the relation

(3.2) $G(rm) = G(n)$ for all n and a given integer $r > 1$.

An infinity of such functions G exists. For we can define $G(m)$ arbitrarily for every m that is not a multiple of r, and then every n that is a multiple of r can be uniquely expressed as

(3.3) $n = m \cdot r^i$ where $r \nmid m$ and $i \geq 1$.

Taking a countable infinity of such functions G and labelling each of them with a unique prime number suffix p, set up a function $F(n)$ defined as

(3.4) $F(n) = \prod_{p|n} G\ (a)$ when $n = \prod p$ in the standard form.

It is easily found that this F satisfies (1.4).

4. We are, in turn, led to finding multiplicative functions that have a multiplicatory-period as defined in (3.2). In such a case

(4.1) $\prod_p G(p^{a+i}) = \prod_p G(p^a),\ n = \prod_p p^a,\ r = \prod_p p^i,$

where p runs through all the primes so that $a, i \geq 0$. Writing $G(p^a)$ as $H_p(a)$, we see that a sufficient condition for (4.1) to hold is that H_p be periodic in a with period i (in the normal sense of periodicity). That is, for every prime p and the corresponding i such that

(4.2) $p^i | r,\ p^{i+1} \nmid r$

we should have

(4.3) $H_p(a + i) = H_p(a)\ \ \forall\, a \in \mathbb{N}$.

A function $H_p(a)$ satisfying (4.2) and (4.3) can be easily constructed by (i) defining $H_p(o)$ as an arbitrary function of the prime argument p and (ii) further defining arbitrary values for $H(a)$ for the values of a in the interval $0 < a < i$, where i is the unique integer corresponding to p given by (4.2). These arbitrary values completely determine the values of $H_p(a)$ for every prime p and every nonnegative integer a, in order that (4.2) and (4.3) hold. Hence, a function G defined by

(4.4) $G(n) = \prod_p H\ (a),\ n = \prod_p p^a,$

where p is a variable prime, is multiplicative and multiplicatory-periodic with n as that period.

5. Special Solutions

The preceding general solution notwithstanding, the particular pairs of functions given in [1] are still of interest. They show how certain simple expressions of known arithmetic functions exhibit the power-periodic relation (1.4), and hence generate friendly-pairs.

The two instances given in [1] actually can be shown to be representatives of two classes of such arithmetic functions.

Write P-periodic for power-periodic, which is the property expressed by (1.4) and M-periodic for the multiplicatory-periodic property expressed in (3.2).

Class I:

Consider the m^{th} root of unity, $\omega = \exp(2\pi i/m)$ for a given $m > 1$. Obviously

(5.1) $k \equiv 1 \pmod{m} \Rightarrow \omega^{kr} = \omega^r$.

That is, ω^r as a function of r is M-periodic with k as an M-period. Construct the multiplicative $f(n)$ defined by its values for powers of primes as $f(p^r) = \omega^r$. Clearly, $f(n) = \omega^{\Omega(n)}$ where $\Omega(n)$ is the total number of prime divisors, repetition reckoned, in the factorization of n. It is also clear that $f(n)$ is P-periodic, with P-period k, i.e., $f(n^k) = f(n)\ \forall\, n \in \mathbb{N}$.

When k happens to be a square, say $k = \alpha^2$, we have

$$f(n^\alpha) = g(n),\ g(n^\alpha) = f(n).$$

In the first friendly-pair given in [1], m is taken as $\alpha + 1$ so that $\alpha \equiv -1 \pmod{m}$, so $\omega^{\alpha r} = \omega^r$ and hence $f(n)g(n) = 1$.

Class II:

The concluding pair of functions given in [1], "friendly" except for the fact that they are not reciprocals of each other, is

$$(5.2)\qquad f(n) = \sum_{dt^3 = n} \mu(d); \quad g(n) = \sum_{d^2t^3 = n} \mu(d)$$

so that

$$(5.3)\qquad f(n^2) = g(n); \quad g(n^2) = f(n); \quad f(n)g(n) = \begin{cases} 1 & \text{if } n \text{ is a cube} \\ 0 & \text{if not.} \end{cases}$$

The summand μ is the Möbius function. The first summation is over divisors d of n such that n/d is a perfect cube. The other summation is over the divisors d of n such that $d^2|n$ and n/d^2 is a perfect cube.

The general class, of which the given example turns out to be representative, is given below.

Take a multiplicative function $c(n)$ that vanishes when n is divisible by an r^{th} power (for a fixed r). There are infinitely many such functions, since $c(p^\lambda)$ can be defined arbitrarily for every prime p and $1 \le \lambda \le r - 1$. Set

$$(5.4)\qquad F(n) = \sum_{dt^r = n} c(d) \quad\text{and}\quad G(n) = \sum_{d^\ell t^r = n} c(d)$$

where r and c are as just assumed and ℓ is any integer such that

$$\exists\, k\colon\ k\ell \equiv 1 \pmod{r}.$$

The summations are over divisors d of n such that n/d is an r^{th} power in the first case and $d^\ell|n$ and n/d^ℓ is an r^{th} power in the second case.

F and G can be proved to be multiplicative. Define

$$(5.5)\qquad T_r(n) = \begin{cases} 1 & \text{if } n \text{ is an } r^{\text{th}} \text{ power} \\ 0 & \text{if not.} \end{cases}$$

Observe that $T_r(n)$ is multiplicative. F and G can now be written as divisor-convolution products.

$$(5.6)\qquad F(n) = \sum_{d|n} c(d)T_r(n/d) \quad\text{and}\quad G(n) = \sum_{d|n} c(d^{1/\ell})T_\ell(d)T_r(n/d),$$

where, in the second summation, c is understood to be zero when $d^{1/\ell}$ is not an integer. Such convolution products of multiplicative functions are multiplicative. Hence, F and G are multiplicative and are consequently characterized by their values for powers of primes. For every prime $p \ge 2$ and $\alpha \ge 1$, we have, by virtue of (5.6),

$$(5.7)\qquad F(p^\alpha) = \sum_{i=0}^{\mathrm{Min}(r-1,\ \alpha)} c(p^i)T\ (p^{\alpha - i}),$$

where "Min" denotes the minimum value from among the arguments within the parentheses. By the nature of the function T_r, it is clear that all the terms but one on the right-hand side of (5.6) have to be zero. The result is that

$$F(p^{\alpha}) = c(p^{\alpha \bmod r}), \tag{5.8}$$

where "α mod r" stands for the remainder left when α is divided by r.

If k and ℓ are two integers such that $k\ell \equiv 1 \pmod{r}$, then

$$F(p^{k\ell\alpha}) = c(p^{k\ell\alpha \bmod r}) = c(p^{\alpha \bmod r}) \tag{5.9}$$

which, by (5.8) $= F(p^{\alpha})$.

Hence

$$F(n^{k\ell}) = \prod_{p|n} F(p^{k\ell\alpha}) \quad \left[\text{where } n = \prod p^{\alpha}\right] \tag{5.10}$$

$$= \prod F(p^{\alpha}) = F(n).$$

That is, F defined in (5.4) is P-periodic, with $k\ell$ for a P-period. So, if we set $F(n^k) = G^*(n)$, then $G^*(n^{\ell}) = F(n)$. We prove below that G^* is the same as G defined in (5.4).

$$F(p^{k\alpha}) = \sum_{i=0}^{\mathrm{Min}(r-1,\, k\alpha)} c(p^i)T_r(p^{k\alpha - i}). \tag{5.11}$$

Now note that

$$T_r(p^{k\alpha - i}) = T_r(p^{k(\alpha - \ell i)}) \tag{5.12}$$

since the indices on both of the sides differ by a multiple of r and T_r is not affected thereby.

Using (5.11) and (5.12), we deduce

$$F(n^k) = \prod_p F(p^{k\alpha}) \quad \text{where } n = \prod p^{\alpha} \text{ in the standard form} \tag{5.13}$$

$$= \prod_p [T_r(p^{k\alpha}) + c(p)T_r(p^{k(\alpha - \ell)}) + c(p^2)T_r(p^{k(\alpha - 2\ell)})$$

$$+ \cdots \text{ until the index on } p \text{ becomes negative}]$$

$$= \sum_{d^{\ell}t^{r} = n} c(d) \quad \text{(multiplied out)}$$

which $= G(n)$ as defined.

6. Three Points and an Open Problem

Before concluding, we make three observations and indicate a promising problem.

Note (i): Pair-wise "friendliness" being found only on off-shoots of power-periodicity, one could study friendly-pairs defined on the basis of M-periodicity and normal periodicity also: Say

$$f(kn) = g(n),\ g(\ell n) = f(n), \text{ so that} \tag{6.1}$$

$$f(k\ell n) = f(n) \quad \text{and} \quad g(k\ell n) = g(n);$$

$$f(n + k) = g(n),\ g(n + \ell) = f(n), \text{ so that} \tag{6.2}$$

$$f(n + k + \ell) = f(n) \quad \text{and} \quad g(n + k + \ell) = g(n).$$

The former of these cases does not appear to be as trivial as the latter, as seen from the construction of M-periodic functions given earlier.

Note (ii): The definitions of P- and M-periodicities, leading to interesting consequences in the case of arithmetic functions, would seem to degenerate into trivialities in the case of functions of a continuous variable.

For instance, defining $f(kx) = f(x)$ for all real x or $f(x^k) = f(x)$ for all real x leads only to f being a constant, if f is to be continuous at zero in the first case and at one in the second case.

Note (iii): Why pairs only? one could ask for r-tuples of functions f_i, $0 \le i \le n - 1$, satisfying the mutual relation.

(6.3) $\quad f_i(n\ (\cdot)\ k_i) = f_{i+1 \bmod r}(n),$

where $(\cdot)$ stands for multiplication or "to the power of." Obviously, every f_i is "$(\cdot)$"-periodic; with $\prod_i k_i$ for a "$(\cdot)$"-period.

Note (iv): In the case of normal periodicity it is well known that if k is a period then there is a divisor of k that is the minimal period (considering arithmetic functions), and a function cannot have more than one fundamental period. That is not true for M- and P-periodic arithmetic functions. It appears promising to study the set of integers

$$\{k^r \ell^s : r,\ s \in \mathbb{N} + \{0\}\}$$

for a given pair of natural numbers k and ℓ.

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Reference

1. N. Balasubramanian & R. Sivaramakrishnan. "Friendly-Pairs of Multiplicative Functions." *Fibonacci Quarterly* 25.4 (1987):320-321.

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