A RESULT ON 1-FACTORS RELATED TO FIBONACCI NUMBERS

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1. Introduction

The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, $F_i = F_{i-1} + F_{i-2}$ for $i \geq 2$. It is well known [3] that the "ladder" composed of n squares (Fig. 1) has F_{n+2} 1-factors.

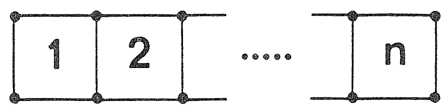

FIGURE 1

A 1-factor of a graph G with $2n$ vertices is a set of n independent edges of G, where independent means that two edges do not have a common endpoint. In the present paper, we investigate the number of 1-factors in a graph $Q_{p,q}$, composed of $p + q + 1$ squares, whose structure is depicted in Figure 2.

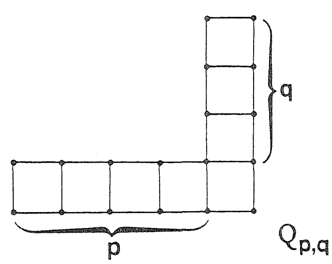

FIGURE 2

Throughout this paper, we assume that the number of squares in $Q_{p,q}$ is fixed and is equal to $n + 1$.

The number of 1-factors of a graph G is denoted by $K\{G\}$.

Lemma 1: $K\{Q_{p,q}\} = F_{n+2} + F_{p+1}F_{q+1}$ where $n = p + q$.

Before proceeding with the proof of Lemma 1 we recall an elementary property of the Fibonacci numbers, which is frequently employed in the present paper:

(1) $$F_m = F_k F_{m-k+1} + F_{k-1}F_{m-k}, \quad 1 \leq k \leq m.$$

Proof: Let the edges of $Q_{p,q}$ be labeled as indicated in Figure 3.

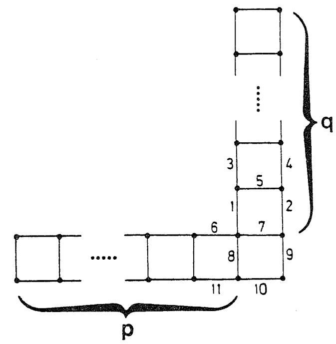

FIGURE 3

First observe that above and below the edges 1 and 2 there is an even number of vertices. Therefore, a 1-factor of $Q_{p,q}$ either contains both the edges 1 and 2 or none of them.

A 1-factor of $Q_{p,q}$ containing the edges 1 and 2 must not contain the edges 3, 4, ..., 9 because they have common endpoints with 1 and/or 2. Then, however, the edge 10 must and the edge 11 must not belong to this 1-factor. The remaining edges of $Q_{p,q}$ form two disconnected ladders with $p - 1$ and $q - 2$ squares, respectively, whose number of 1-factors is evidently $F_{p+1}F_q$. Therefore, there are $F_{p+1}F_q$ 1-factors of $Q_{p,q}$ containing the edges 1 and 2.

The edges of $Q_{p,q}$ without 1 and 2 form two disconnected ladders with $p + 1$ and $q - 1$ squares, respectively. Consequently, there are $F_{p+3}F_{q+1}$ 1-factors of $Q_{p,q}$ which do not contain the edges 1 and 2.

This gives

$$K\{Q_{p,q}\} = F_{p+3}F_{q+1} + F_{p+1}F_q = F_{p+2}F_{q+1} + F_{p+1}F_{q+1} + F_{p+1}F_q$$
$$= F_{p+q+2} + F_{p+1}F_{q+1},$$

where the identity (1) was used. Lemma 1 follows from the fact that $p + q = n$. □

2. Minimum and Maximum Values of $K\{Q_{p,q}\}$

Theorem 1: The minimum value of $K\{Q_{p,q}\}$, $p + q = n$, is achieved for $p = 1$ or $q = 1$.

Proof: Bearing in mind Lemma 1, it is sufficient to demonstrate that for $0 \le p \le n$,

$$F_2F_n \le F_{p+1}F_{n-p+1},$$

with equality if and only if $p = 1$ or $p = n - 1$.

Now, using (1),

$$F_{p+1}F_{n-p+1} = F_{n+1} - F_pF_{n-p} = F_n + F_{n-1} - F_pF_{n-p}$$
$$= F_n + F_pF_{n-p} + F_{p-1}F_{n-p-1} - F_pF_{n-p}$$
$$= F_n + F_{p-1}F_{n-p-1} \ge F_n = F_2F_n.$$

Because of $F_0 = 0$, equality in the above relation occurs if and only if $p - 1 = 0$ or $n - p - 1 = 0$. □

Theorem 2: The maximum value of $K\{Q_{p,q}\}$, $p + q = n$, is achieved for $p = 0$ or $q = 0$.

Proof:

$$F_{p+1}F_{n-p+1} = F_{n+1} - F_pF_{n-p} \le F_{n+1} = F_1F_{n+1}$$

with equality if and only if $p = 0$ or $p = n$. Theorem 2 follows now from Lemma 1. □

Theorem 3: If $p \ne 0$, $q \ne 0$, then the maximum value of $K\{Q_{p,q}\}$, $p + q = n$, is achieved for $p = 2$ or $q = 2$.

Proof: From the proof of Theorem 1 we know that for $0 < p < n$,

$$F_2F_{n-2} \le F_pF_{n-p}$$

with equality for $p = 2$ or $p = n - 2$. This inequality implies

$$F_{n+1} - F_2F_{n-2} \ge F_{n+1} - F_pF_{n-p},$$

i.e.,

$$F_3F_{n-1} + F_2F_{n-2} - F_2F_{n-2} \ge F_{p+1}F_{n-p+1} + F_pF_{n-p} - F_pF_{n-p},$$

i.e.,

$$F_3F_{n-1} \ge F_{p+1}F_{n-p+1},$$

from which Theorem 3 follows immediately. □

Theorem 4: If $p \ne 1$, $q \ne 1$, then the minimum value of $K\{Q_{p,q}\}$, $p + q = n$, is achieved for $p = 3$ or $q = 3$.

Proof: We start with the inequality

$$F_3F_{n-3} \ge F_pF_{n-p}$$

which was deduced within the proof of Theorem 3 and in a fully analogous manner obtain

$$F_4F_{n-2} \le F_{p+1}F_{n-p+1}$$

with equality for $p + 1 = 4$ or $p + 1 = n - 2$. □

3. The Main Result

The reasoning employed to prove Theorems 3 and 4 can be further continued, leading ultimately to the main result of the present paper.

Theorem 5:

(a) If n is odd, then

$$K\{Q_{0,n}\} > K\{Q_{2,n-2}\} > K\{Q_{4,n-4}\} > \cdots > K\{Q_{n-3,3}\} > K\{Q_{n-1,1}\}.$$

(b) If n is even and divisible by four, then

$$K\{Q_{0,n}\} > K\{Q_{2,n-2}\} > \cdots > K\{Q_{n/2,\,n/2}\} > K\{Q_{n/2+1,\,n/2-1}\}$$
$$> K\{Q_{n/2+3,\,n/2-3}\} > \cdots > K\{Q_{n-3,3}\} > K\{Q_{n-1,1}\}.$$

(c) If n is even, but not divisible by four, then

$$K\{Q_{0,n}\} > K\{Q_{2,n-2}\} > \cdots > K\{Q_{n/2-1,\,n/2+1}\} > K\{Q_{n/2,\,n/2}\}$$
$$> K\{Q_{n/2+2,\,n/2-2}\} > \cdots > K\{Q_{n-3,3}\} > K\{Q_{n-1,1}\}.$$

All the above inequalities are strict.

4. Discussion and Applications

There seem to be many ways by which the present results can be extended and generalized. It is easy to see that if in the graph $Q_{p,q}$ some (or all) structural details of the type A and B are replaced by A^* and B^*, respectively (see Fig. 4), the number of 1-factors will remain the same. This means that our results hold also for chains of hexagons. In particular, it is long known

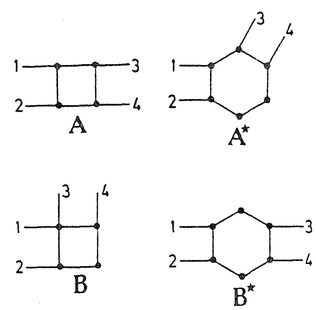

FIGURE 4

[2] that the zig-zag chain of n hexagons (Fig. 5) has F_{n+2} 1-factors. As a matter of fact, the number of 1-factors of chains of hexagons are of some importance in theoretical chemistry [1] and quite a few results connected with Fibonacci numbers have been obtained in this field (see [1] and the references cited therein).

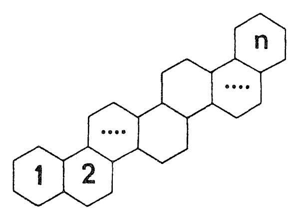

FIGURE 5

References

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