FIBONACCI HYPERBOLAS

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1. Introduction

Is it possible for a hyperbola $h(x, y) = 0$ to pass through infinitely many points of the form (F_m, F_n), whose coordinates are distinct Fibonacci numbers? The answer to this question is yes. For example, the hyperbola $x^2 + xy - y^2 + 1 = 0$ passes through the points (1, 2), (3, 5), (8, 13), (21, 34), (55, 89), ...

It is not difficult to discover other hyperbolas

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

that pass through infinitely many (F_m, F_n). We shall call such a hyperbola a *Fibonacci hyperbola*. Bergum [1] and Horadam [2] have discussed classes of conic sections that include Fibonacci hyperbolas. In particular, formulas (1) and (1′) below occur, after substitutions, among those discussed by Bergum and Horadam. The purpose of this note is to prove that these formulas account for *all* the Fibonacci hyperbolas. There are no others.

2. Formula, Examples, and Graphs

As usual, let $F_0, F_1, F_2, F_3, F_4, F_5, F_6, \ldots$ denote the Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, ..., and let $L_0, L_1, L_2, L_3, L_4, L_5, L_6, \ldots$ denote the Lucas sequence 2, 1, 3, 4, 7, 11, 18, ... We extend these sequences in the usual way:

$$F_n = (-1)^{n+1}F_{-n} \quad \text{and} \quad L_n = (-1)^n L_{-n}, \quad \text{for } n = -1, -2, -3, \ldots .$$

It will be helpful to list the first few hyperbolas of the form

$$p_n(x, y) = x^2 + (-1)^{n+1}L_n xy + (-1)^n y^2 + F_n^2 = 0, \quad \text{for } n = 1, 2, 3, \ldots, \tag{1}$$

along with representative points that lie on each hyperbola:

TABLE 1

Hyperbola	Representative Points
$p_1(x, y) = x^2 + xy - y^2 + 1 = 0$	(1, 2), (3, 5), (8, 13), (21, 34), (55, 89)
$p_2(x, y) = x^2 - 3xy + y^2 + 1 = 0$	(1, 2), (2, 5), (5, 13), (13, 34), (34, 89)
$p_3(x, y) = x^2 + 4xy - y^2 + 4 = 0$	(1, 5), (3, 13), (8, 34), (21, 89), (55, 233)
$p_4(x, y) = x^2 - 7xy + y^2 + 9 = 0$	(1, 5), (2, 13), (5, 34), (13, 89), (34, 233)
$p_5(x, y) = x^2 + 11xy - y^2 + 25 = 0$	(1, 13), (3, 34), (8, 89), (21, 233), (55, 610)
$p_6(x, y) = x^2 - 18xy + y^2 + 64 = 0$	(1, 13), (2, 34), (5, 89), (13, 233), (34, 610)

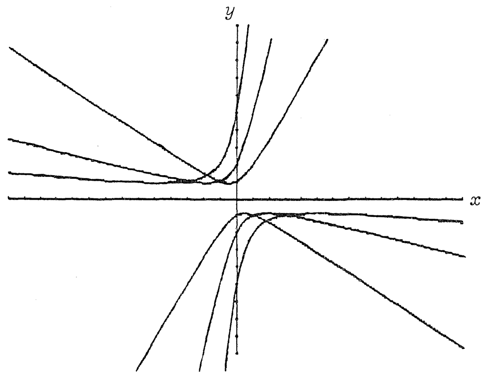

FIGURE 1

The Fibonacci hyperbolas
$p_1(x, y)$, $p_3(x, y)$, and $p_5(x, y)$

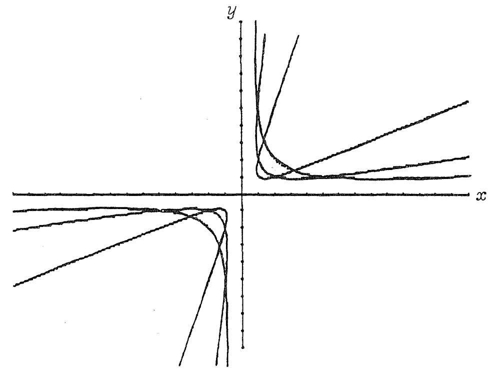

FIGURE 2

The Fibonacci hyperbolas
$p_2(x, y)$, $p_4(x, y)$, and $p_6(x, y)$

Theorem 1: Each hyperbola of the form

(1) $\quad p_n(x, y) = x^2 + (-1)^{n+1}L_n xy + (-1)^n y^2 + F_n^2 = 0$, for $n = 1, 2, 3, \ldots$,

is a Fibonacci hyperbola.

Proof: Well-known identities, given as I_{24} and I_{19} in Hoggatt [3], show that for odd n and even m,

$$\begin{aligned} F_m^2 + (L_n F_m - F_{n+m})F_{n+m} + F_n^2 &= F_m - F_{n-m}F_{n+m} + F_n^2 \\ &= F_m^2 + F_n^2 - [F_n^2 + (-1)^{m+n+1}F_m^2] \\ &= 0. \end{aligned}$$

Similarly, identities I_{22} and I_{19} yield analogous results for even n and odd m. Thus, for any positive integer n, positive even integer h, and integer k for which $k + n$ is odd, all the points

$$(F_k, F_{k+n}), (F_{k+h}, F_{k+n+h}), (F_{k+2h}, F_{k+n+2h}), \ldots$$

lie on hyperbola (1).

Theorem 2: Each hyperbola of the form

(1′) $\quad q_n(x, y) = x^2 + (-1)^{n+1}L_n xy + (-1)^n y^2 - F_n^2 = 0$, for $n = 1, 2, 3, \ldots$,

is a Fibonacci hyperbola.

Proof: For odd n and odd m, identities I_{22} and I_{19} yield

$$\begin{aligned} F_m^2 + (L_n F_m - F_{n+m})F_{n+m} - F_n^2 &= F_m^2 + F_{n-m}F_{n+m} - F_n^2 \\ &= F_m^2 - F_n^2 + (F_n^2 - F_m^2) \\ &= 0. \end{aligned}$$

Similarly, $q_n(F_m, F_{n+m}) = 0$ for even n and even m. As in the proof of Theorem 1, it now follows that for any positive integer n, positive even integer h, and integer k for which $k + n$ is even, all the points

$$(F_k, F_{k+n}), (F_{k+h}, F_{k+n+h}), (F_{k+2h}, F_{k+n+2h}), \ldots$$

lie on hyperbola (1′).

TABLE 2

Hyperbola	Representative Points
$q_1(x, y) = x^2 + xy - y^2 - 1 = 0$	(1, 1) (2, 3), (5, 8), (13, 21), (34, 55)
$q_2(x, y) = x^2 - 3xy + y^2 - 1 = 0$	(1, 3), (3, 8), (8, 21), (21, 55), (55, 144)
$q_3(x, y) = x^2 + 4xy - y^2 - 4 = 0$	(1, 3), (2, 8), (5, 21), (13, 55), (34, 144)
$q_4(x, y) = x^2 - 7xy + y^2 - 9 = 0$	(1, 8), (3, 21), (8, 55), (21, 144), (55, 377)
$q_5(x, y) = x^2 + 11xy - y^2 - 25 = 0$	(1, 8), (2, 21), (5, 55), (13, 144), (34, 377)
$q_6(x, y) = x^2 - 18xy + y^2 - 64 = 0$	(1, 21), (3, 55), (8, 144), (21, 377), (55, 987)

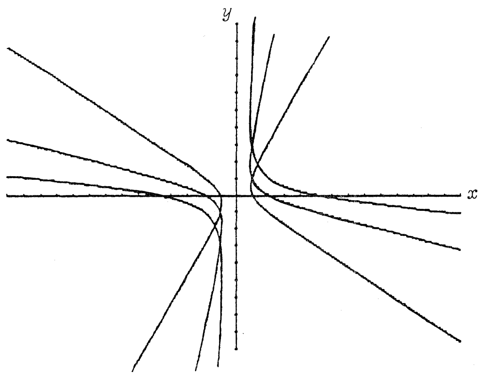

FIGURE 3

The Fibonacci polynomials $q_1(x, y)$, $q_3(x, y)$, and $q_5(x, y)$

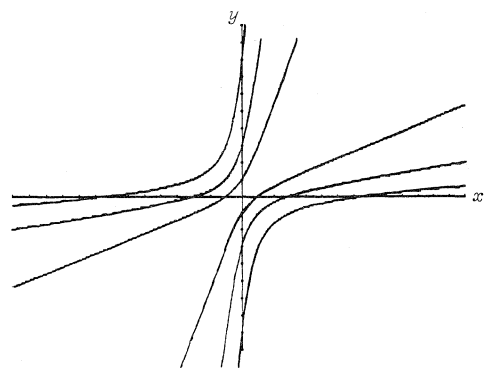

FIGURE 4

The Fibonacci polynomials $q_2(x, y)$, $q_4(x, y)$, and $q_6(x, y)$

3. The Main Theorem

In this section we shall state and prove the main theorem of this paper, which expresses every Fibonacci hyperbola in terms of the polynomials $p_n(x, y)$ and $q_n(x, y)$.

Lemma 3.1: The coefficients a, b, c, d, e, f in the equation

(2) $$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

of a Fibonacci hyperbola can be chosen to be integers. (Following the proof of this lemma, these coefficients will be understood to be integers except where stated otherwise.)

Proof: Divide both sides of (2) by one of the nonzero coefficients, and then substitute for (x, y) any five distinct (F_m, F_n) that lie on the hyperbola. Cramer's Rule applied to the resulting five equations shows that each coefficient is a rational number. Let D be the least common multiple of the five denominators. Write (2) using the five rational numbers and 1 as coefficients, and then multiply both sides by D. The resulting coefficients are integers.

Lemma 3.2: Suppose (2) is a hyperbola that passes through the points (F_{s_n}, F_{t_n}) for some pair $s_1, s_2, s_3, \ldots$ and $t_1, t_2, t_3, \ldots$ of nondecreasing sequences of integers. Then there exist constants m and N such that $t_n - s_n = m$ for all $n > N$.

Proof: The proof will be in three cases.

Case 1. Suppose $c = 0$. Then $b \neq 0$, else (2) would not represent a hyperbola. Divide both sides of (2) by x^2 to find

$$\begin{aligned} -a/b &= \lim_{x\to\infty} y/x = \lim_{n\to\infty} F_{t_n}/F_{s_n} \\ &= \lim_{n\to\infty}(\alpha^{t_n} - \beta^{t_n})/(\alpha^{s_n} - \beta^{s_n}) = \lim_{n\to\infty} \alpha^{t_n - s_n}, \end{aligned}$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$.

If $a = 0$, then $\lim_{x\to\infty} u = -d/b$, so that $\lim_{n\to\infty} F_{t_n} = -d/b$, which is impossible, and so $a \neq 0$. Consequently $\lim_{n\to\infty} \alpha^{t_n - s_n}$ is a nonzero constant. The exponent $t_n - s_n$ is an integer for all n, so that $t_n - s_n$ is a constant for all sufficiently large n.

Case 2. If $c \neq 0$ and $a = 0$, then $b \neq 0$, else (2) would not represent a hyperbola. Divide both sides of (2) by y^2 to find

$$-c/b = \lim_{y\to\infty} x/y = \lim_{n\to\infty} F_{s_n}/F_{t_n} = \lim_{n\to\infty} \alpha^{s_n - t_n},$$

so that $s_n - t_n$, and hence $t_n - s_n$ is a constant for all sufficiently large n.

Case 3. If $c \neq 0$ and $a \neq 0$, then divide both sides of (2) by x^2 and solve the resulting equation for y/x to obtain the slopes of the asymptotes:

$$(-b \pm \sqrt{b^2 - 4ac})/2c = \lim_{x\to\infty} y/x = \lim_{n\to\infty} F_{t_n}/F_{s_n} = \lim_{n\to\infty} \alpha^{t_n - s_n},$$

so that $t_n - s_n$ must be a constant for all sufficiently large n.

Theorem 3: For $n = 1, 2, \ldots$, let

$$p_n(x, y) = x^2 + (-1)^{n+1}L_n xy + (-1)^n y^2 + F_n^2 = 0,$$

and let $q_n(x, y) = p_n(x, t) - 2F_n^2$. Every Fibonacci polynomial is one of the following forms:

$$p_n(x, y) = 0, \quad q_n(x, y) = 0, \quad p_n(-x, y) = 0, \quad \text{or} \quad q_n(-x, y) = 0.$$

Proof: Suppose

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \tag{2}$$

is a hyperbola that passes through the points (F_{s_n}, F_{t_n}) for some pair $s_1, s_2, s_3, \ldots$ and $t_1, t_2, t_3, \ldots$ of nondecreasing sequences of integers. We refer to the three cases of Lemma 3.2 and show first that Case 1 and Case 2 cannot occur. Let m be as in Lemma 3.2; note that m can be negative.

In Case 1, if $m = 0$, then infinitely many (F_{s_n}, F_{s_n}) lie on the conic section (2), and so (2) represents the line $y = x$, not a hyperbola. If $m = 0$, then the equation

$$-a/b = \alpha^m = [(1 + \sqrt{5})/2]^m = (L_m + \sqrt{5}F_m)/2$$

shows that a/b is not a rational number, contrary to Lemma 3.1. We conclude that $c \neq 0$.

In Case 2, $-c/b = \alpha^{t_n - s_n}$, an irrational constant for all sufficiently large n, contrary to Lemma 3.1. Consequently, $c \neq 0$ and $a \neq 0$, which is Case 3.

In Case 3, $(-b \pm \sqrt{b^2 - 4ac})/2c = \alpha^m = (L_m + \sqrt{5}F_m)/2$. Separating rational and irrational parts yields

(3) $\qquad -b/c = L_m$ and $\pm(\sqrt{b^2 - 4ac})/c = \sqrt{5}F_m$.

Substitute $-cL_m$ for b into the second equation and obtain

$$c = 4a/(L_m^2 - 5F_m^2) = (-1)^m a.$$

We may and do assume that $a = 1$ (allowing d, e, f to be rational numbers), so that (2) takes the form

(4) $\qquad x^2 - (-1)^m L_m xy + (-1)^m y^2 + dx + ey + f = 0,\ m = \pm 1, \pm 2, \pm 3, \ldots$.

Now, substitute (F_{s_n}, F_{s_n+m}) for (x, y) into (4):

$$F_{s_n}^2 - (-1)^m L_m F_{s_n} F_{s_n+m} + (-1)^m F_{s_n+m}^2 + dF_{s_n} + eF_{s_n+m} + f = 0.$$

Using identities I_{21} (if m is even) and I_{23} (if m is odd) from [3] gives

$$(-1)^m [F_{s_n+m}^2 - F_{s_n} F_{s_n+2m}] + dF_{s_n} + eF_{s_n+m} + f = 0.$$

Identity I_{19} from [3] then gives

$$(-1)^{s_n+m} F_m^2 + dF_{s_n} + eF_{s_n+m} + f = 0.$$

Let n_1, n_2, n_3 be any three integers, exceeding N, for which the three integers s_{n_1}, s_{n_2}, s_{n_3} are either all odd or all even. Then the system

(5) $\qquad dF_{s_{n_i}} + eF_{s_{n_i}+m} + f = (-1)^{s_{n_i}+m+1} F_m^2$, for $i = 1, 2, 3$,

has the unique solution

$$d = 0,\ e = 0,\ f = (-1)^{s_{n_i}+m+1} F_m^2.$$

Clearly, the s_{n_i}, for $i = 1, 2, 3, \ldots$, must all be odd or must all be even, else the infinite system (5) has no solution.

Case 1. Suppose the s_{n_i} are all odd. Rewrite (4) as

(6) $\qquad x^2 - (-1)^m L_m xy + (-1)^m y^2 + (-1)^m F_m = 0,\ m = \pm 1, \pm 2, \pm 3, \ldots$.

If $m < 0$, then $L_m = (-1)^m L_{-m}$ and $F_m = (-1)^{m+1} F_{-m}$, so that

$$x^2 - L_{-m} xy + (-1)^m y^2 + (-1)^m F_{-m}^2 = 0.$$

Substitute n for $-m$ to obtain

(7) $\qquad x^2 - L_n xy + (-1)^n y^2 + (-1)^n F_n^2 = 0.$

If n is even, (7) is $p_n(x, y) = 0$; if n is odd, (7) is $q_n(-x, y) = 0$.

If $n = m > 0$ and is even, then (6) is $p_n(x, y) = 0$. If $n = m > 0$ is odd, then (6) is $q_n(x, y) = 0$.

Case 2. Suppose the s_{n_i} are all even. Rewrite (4) as

(6′) $\qquad x^2 - (-1)^m L_m xy + (-1)^m y^2 - (-1)^m F_m^2 = 0,\ m = \pm 1, \pm 2, \pm 3, \ldots$.

If $m < 0$, then write $n = -m$, so that

(7′) $\qquad x^2 - L_n xy + (-1)^n y^2 - (-1)^n F_n^2 = 0.$

If n is even, (7′) is $q_n(x, y) = 0$; if n is odd, (7′) is $q\ (-x, y) = 0$.

If $n = m > 0$ and is even, then (6′) is $q_n(x, y) = 0$. If $n = m > 0$ is odd, then (6′) is $p_n(x, y) = 0$.

4. Concluding Remarks

Theorem 3 establishes the following representation for all Fibonacci hyperbolas:

$$y^2 + bxy + (-1)^n x^2 + f = 0, \quad \text{where } |b| = L_n \text{ and } |f| = F_n^2.$$

Each of these hyperbolas consists of two branches:

$$y = (-bx + \sqrt{[b^2 - 4(-1)^n]x^2 - 4f})/2$$

and

$$y = (-bx - \sqrt{[b^2 - 4(-1)^n]x^2 - 4f})/2.$$

The representative points listed in Tables 1 and 2 lie on the *upper* branch of their respective hyperbolas. Does the *lower* branch also pass through points that are closely associated with Fibonacci numbers? The affirmative answer to this question follows from Bergum [1, pp. 27-28].

References

1. Gerald E. Bergum. "Addenda to Geometry of a Generalized Simson's Formula." *Fibonacci Quarterly 22.1* (1984):22-28.
2. A. F. Horadam. "Geometry of a Generalized Simson's Formula." *Fibonacci Quarterly 20.2* (1982):164-168.
3. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969.

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