THIRD-ORDER DIAGONAL FUNCTIONS OF PELL POLYNOMIALS

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1. Introduction

This paper is concerned with the study of some third-order sequences of polynomials. While it is only of an introductory nature, it does give something of the flavor of the research involved. In particular, we have found that an examination of the roots of the auxiliary equation to be a challenging and rewarding endeavor.

The first of these sequences is $\{r_n(x)\}$. It is defined thus:

$$(1.1)\quad \begin{cases} r_0(x) = 0,\ r_1(x) = 1,\ r_2(x) = 2x \\ r_{n+1}(x) = 2xr_n(x) + r_{n-2}(x) \quad (n \geq 2) \end{cases}$$

Two other sequences, namely $\{s_n(x)\}$ and $\{t_n(x)\}$, are also considered. They are defined thus:

$$(1.2)\quad \begin{cases} s_0(x) = 0,\ s_1(x) = 2,\ s_2(x) = 2x \\ s_{n+1}(x) = 2xs_n(x) + s_{n-2}(x) \quad (n \geq 2) \end{cases}$$

$$(1.3)\quad \begin{cases} t_0(x) = 3,\ t_1(x) = 2x,\ t_2(x) = 4x^2 \\ t_{n+1}(x) = 2xt_n(x) + t_{n-2}(x) \quad (n \geq 2) \end{cases}$$

These sequences are called *third-order diagonal functions of Pell polynomials* [5], or simply *Pell diagonal functions*, because the first two coincide with sequences derived by taking the "diagonals" of gradient 1 from the arrays produced by Pell and Pell-Lucas polynomials [10].

The three sequences can be considered to be constructed from the diagonals of gradient 2 from the arrays produced by expansions of

$$(2x + 1)^n,\quad (2x + 2)(2x + 1)^{n-1},\quad (2x + 3)(2x + 1)^{n-1},$$

where $n \geq 1$.

Considered as a sequence of order three, $\{s_n(x)\}$ appears to be of little significance. The sequences $\{r_n(x)\}$ and $\{t_n(x)\}$ may be deemed to be the *fundamental* and *primordial* sequences, respectively, for those obeying the recurrence relation in (1.1)-(1.3) [9]. All of these sequences are too special to provide subject matter for the study of third-order sequences in general. In a later paper some generalizations of these polynomials may be considered and these are closer to typical third-order sequences.

Jaiswal [6] and Horadam [4] studied the diagonal functions of Chebyshev polynomials of the second and first kinds, respectively, $\{p_n(x)\}$ and $\{q_n(x)\}$. It may be shown that

$$(1.4)\quad \begin{cases} r_n(x) = (-1)^{n-1}p_n(-x), & r_n(-x) = (-1)^{n-1}p_n(x) \\ s_n(x) = (-1)^{n-1}q_n(-x), & s_n(-x) = (-1)^{n-1}q_n(x) \end{cases}$$

Simple relations such as these are to be expected as Pell and Pell-Lucas sequences are complex Chebyshev polynomials [5].

2. Roots of the Auxiliary Equation of the Pell Diagonal Functions

The auxiliary equation of the diagonal functions (1.1)-(1.3) is the cubic

$$(2.1)\quad f(y) \equiv y^3 - 2xy^2 - 1 = 0.$$

By Descartes' Rule, one of the roots is real and positive. Denote this by α. For $x \geq 0$, the other two roots, β and γ are conjugate complex numbers. It is noted that, from (2.1),

$$(2.1')\quad \begin{cases} \alpha + \beta + \gamma = 2x \\ \alpha\beta + \beta\gamma + \gamma\alpha = 0 \\ \alpha\beta\gamma = 1 \end{cases}$$

By using Cardano's procedure [3], it is found that

$$(2.2)\quad \begin{cases} \alpha = 2x/3 + \sqrt[3]{\{16x^3 + 27 + \sqrt{(864x^3 + 729)}\}}/3\sqrt[3]{2} \\ \qquad + \sqrt[3]{\{16x^3 + 27 - \sqrt{(864x^3 + 729)}\}}/3\sqrt[3]{2} \\ \beta = 2x/3 + \omega\sqrt[3]{\{16x^3 + 27 + \sqrt{(864x^3 + 729)}\}}/3\sqrt[3]{2} \\ \qquad + \omega^2\sqrt[3]{\{16x^3 + 27 - \sqrt{(864x^3 + 729)}\}}/3\sqrt[3]{2} \\ \gamma = 2x/3 + \omega^2\sqrt[3]{\{16x^3 + 27 + \sqrt{(864x^3 + 729)}\}}/3\sqrt[3]{2} \\ \qquad + \omega\sqrt[3]{\{16x^3 + 27 - \sqrt{(864x^3 + 729)}\}}/3\sqrt[3]{2} \end{cases}$$

where ω and ω^2 are complex cube roots of unity; α, β, and γ are clearly algebraic functions of x. We use function notation with the roots where appropriate. From (2.2), it is seen that, for $x > -3/2\sqrt[3]{4}$, the quantities

$$(2.3)\quad \begin{cases} u = \sqrt[3]{\{16x^3 + 27 + \sqrt{(864x^3 + 729)}\}} \\ v = \sqrt[3]{\{16x^3 + 27 - \sqrt{(864x^3 + 729)}\}} \end{cases}$$

are real and so β and γ are conjugate complex numbers. If

$$(2.3')\quad x = -3/2\sqrt[3]{4} = d, \text{ then } \beta = \gamma.$$

Again from (2.2), it may be shown that

$$(2.4)\quad \alpha^2 - |\beta|^2 = 2x\alpha \text{ for } x > d.$$

Hence

$$(2.5)\quad \begin{cases} \alpha > |\beta| = |\gamma| \text{ for } x > 0 \\ \alpha = |\beta| = |\gamma| = 1 \text{ for } x = 0 \\ \alpha < |\beta| = |\gamma| \text{ for } d < x < 0. \end{cases}$$

For $x < d$, it is convenient to consider the roots to be given by

$$(2.6)\quad \begin{cases} \alpha = 2x/3 + 4x/3\cos(4\pi + \theta)/3 \\ \beta = 2x/3 + 4x/3\cos(\theta/3) \\ \gamma = 2x/3 + 4x/3\cos(2\pi + \theta)/3 \end{cases}$$

where

$$(2.7)\quad \cos\theta = (16x^3 + 27)/16x^3, \quad \sin\theta = 3\sqrt{\{3(32x^3 + 27)\}}/16ix^3 = 3\sqrt{(3D)}/16x^3$$

D being the discriminant of (2.1), and thus

$$(2.8)\quad D = -(32x^3 + 27).$$

It may be shown that, for $x < d$,

$$(2.9)\quad \begin{cases} -\pi < \theta < 0 \\ \alpha > 0 \\ \beta,\ \gamma < 0 \\ |\beta| > |\gamma| > \alpha \\ |\beta| > 1 \\ |\gamma| > 1 \text{ for } -1 < x < d \\ |\gamma| < 1 \text{ for } x < -1 \\ \lim_{x\to-\infty} \theta = 0^- \\ \lim_{x\to-\infty} \alpha = 0^+ \\ \lim_{x\to-\infty} \beta = -\infty \\ \lim_{x\to-\infty} \gamma = 0^- \end{cases}$$

Some simple correspondences for x, θ, α, β, and γ are recorded in Table 2.1.

TABLE 2.1

x	θ	α	β	γ
d	$-\pi$	$1/\sqrt[3]{4}$	$-2/\sqrt[3]{4}$	$-2/\sqrt[3]{4}$
-1	$-\cos^{-1}(-11/16)$	$(\sqrt{5} - 1)/2$	$-(\sqrt{5} + 1)/2$	-1
$-3/2\sqrt[3]{2}$	$-\pi/2$	$(\sqrt{3} - 1)/\sqrt[3]{2}$	$-(\sqrt{3} + 1)/\sqrt[3]{2}$	$-1/\sqrt[3]{2}$
$-\infty$	0	0	$-\infty$	0

A computer investigation carried out by Br. V. Cotter indicates that, in the natural domain, α is an increasing function, that $|\beta|$ is a decreasing function, and that $|\gamma|$ increases, reaches a maximum near $x = d$ and then decreases to zero.

It is noted that $\alpha(-1)$ and $\beta(-1)$ are negatives of the roots of the auxiliary equation of the Fibonacci sequence. As a result, we would expect that there are simple relations between $\{r_n(-1)\}$, $\{s_n(-1)\}$, and $\{t_n(-1)\}$ and the Fibonacci and Lucas sequences. In fact, a study of the diagonal functions has resulted in obtaining what appears to be a large number of highly specific identities for these numbers. We were alerted to these possibilities by the work of Jaiswal [6] and Horadam [4] dealing with the diagonal functions of the Chebyshev polynomials.

3. Binet Formulas for the Diagonal Functions

A variety of procedures may be followed to give a number of formulas for the diagonal functions in terms of the roots, α, β, and γ. It may be shown that, for $\beta \neq \gamma$,

$$(3.1) \qquad r_n(x) = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^{n+1} & \beta^{n+1} & \gamma^{n+1} \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} = \Delta_{n+1}(x)/\Delta_2(x)$$

$$(3.2) \qquad r_n(x) = \begin{vmatrix} 1 & 1 & 1 \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha^{n+3} & \beta^{n+3} & \gamma^{n+3} \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 & 1 \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha & \beta & \gamma \end{vmatrix} = \delta_{n+3}(x)/\delta_1(x)$$

$$(3.3) \qquad r_n(x) = A_\alpha^n + B_\beta^n + C_\gamma^n$$

where

$$(3.4) \qquad A = \frac{\alpha}{(\alpha - \beta)(\alpha - \gamma)}, \quad B = \frac{\beta}{(\beta - \gamma)(\beta - \alpha)}, \text{ and } C = \frac{\gamma}{(\gamma - \alpha)(\gamma - \beta)}$$

and

$$(3.5) \qquad r_n(x) = \frac{\alpha^{n+1}}{f'(\alpha)} + \frac{\beta^{n+1}}{f'(\beta)} + \frac{\gamma^{n+1}}{f'(\gamma)}$$

where $f(y)$ is as defined in (2.1).

The formula (3.1) may be considered to be the third-order analogue of the Binet formula for the Fibonacci numbers expressed as the quotient of two determinants. The third-order number sequence equivalents of (3.3) and (3.4) occur in Jarden [7] and Spickerman [11] and (3.5) may be compared to a formula of Levesque [8].

Starting with (3.1), we can deduce (1.1). Hence (3.1) could be taken as the definition of $\{r_n(x)\}$. This new definition would allow us to introduce negative subscripts.

Binet formulas for $\{s_n(x)\}$ include, for $\beta \neq \gamma$,

$$(3.6) \qquad s_n(x) = A'\alpha^n + B'\beta^n + C'\gamma^n$$

where

$$(3.7) \qquad A' = \frac{\alpha - \beta - \gamma}{(\alpha - \beta)(\alpha - \gamma)}, \quad B' = \frac{\beta - \alpha - \gamma}{(\beta - \gamma)(\beta - \alpha)}, \text{ and } C' = \frac{\gamma - \beta - \alpha}{(\gamma - \alpha)(\gamma - \beta)}$$

and

$$(3.8) \qquad s_n(x) = \frac{\alpha^{n+1} + \alpha^{n-2}}{f'(\alpha)} + \frac{\beta^{n+1} + \beta^{n-2}}{f'(\beta)} + \frac{\gamma^{n+1} + \gamma^{n-2}}{f'(\gamma)}.$$

The Binet formula for $t_n(x)$ is

$$(3.9) \qquad t_n(x) = \alpha^n + \beta^n + \gamma^n.$$

It may be shown that when $\beta = \gamma$, i.e., when $x = d$,

$$(3.10) \qquad r_n(d) = (-1)^{n-1}2^{(2-2n)/3}\{(3n + 1)2^n - (-1)^n\}/9.$$

The Binet formulas lead to some simple identities involving the diagonal functions, for example,

(3.11) $s_n(x) = r_n(x) + r_{n-3}(x) = 2xr_{n-1}(x) + 2r_{n-3}(x);$

(3.12) $t_{n-1}(x) = r_n(x) + 2r_{n-3}(x) = 2xr_{n-1}(x) + 3r_{n-3}(x) = s_n(x) + r_{n-3}(x).$

The formulas also give the following relations with the Fibonacci and Lucas numbers, $\{f_n\}$ and $\{\ell_n\}$:

$$(3.13)\quad \begin{cases} r_n(-1) = (-1)^{n+1}(f_{n+2} - 1) \\ r_{-n}(-1) = f_{n-2} + (-1)^n \\ s_n(-1) = 2(-1)^{n-1}f_n \\ s_{-n}(-1) = 2f_n \\ t_n(-1) = (-1)^n(\ell_n + 1) \\ t_{-n}(-1) = \ell_n + (-1)^n \end{cases}$$

4. Determinantal Generators for the Diagonal Functions

Let us now introduce a new sequence $\{\phi_n(x)\}$ of determinants of which the first few members are:

$$\phi_1(x) = |2x|,\ \phi_2(x) = \begin{vmatrix} 2x & 1 \\ 0 & 2x \end{vmatrix},\ \phi_3(x) = \begin{vmatrix} 2x & 1 & 0 \\ 0 & 2x & 1 \\ 1 & 0 & 2x \end{vmatrix}$$

The n^{th} term is defined thus:

$$(4.1)\quad \phi_n(x): \left\{\begin{array}{lll} d_{rr} = 2x & \text{for } r = 1, 2, \ldots, n \\ d_{r,r+1} = 1 & \text{for } r = 1, 2, \ldots, n-1 \\ d_{r,r-2} = 1 & \text{for } r = 3, 4, \ldots, n \\ d_{rc} = 0 & \text{otherwise} \end{array}\right\}$$

where d_{rc} is the entry in the r^{th} row and c^{th} column. It may be proved by induction that, for $n > 0$,

(4.2) $\phi_n(x) = r_{n+1}(x).$

The sequences $\{\phi_n^*(x)\}$, $\{\phi_n^{**}(x)\}$ are defined similarly, except that

$d_{12} = 2, 3$, respectively.

Induction shows that, for $n > 0$,

(4.3) $\phi_n^*(x) = s_{n+1}(x);$

(4.4) $\phi_n^{**}(x) = t_n(x).$

Next we introduce a further sequence $\{\eta_n(x)\}$ of which the first few members are:

$$\eta_1(x) = |0|,\ \eta_2(x) = \begin{vmatrix} 0 & 1 \\ 2x & 0 \end{vmatrix},\ \eta_3(x) = \begin{vmatrix} 0 & 1 & 0 \\ 2x & 0 & 1 \\ 1 & 2x & 0 \end{vmatrix}$$

The n^{th} term is specified thus:

$$(4.5)\qquad \eta_n(x):\ \begin{Bmatrix} d_{r,r+1} = 1 & \text{for } r = 1, 2, \ldots, n-1 \\ d_{r,r-1} = 2x & \text{for } r = 2, 3, \ldots, n \\ d_{r,r-2} = 1 & \text{for } r = 3, 4, \ldots, n \\ d_{rc} = 0 & \text{otherwise} \end{Bmatrix}$$

Induction may be employed to prove that

$$(4.6)\qquad \eta_n(x) = r_{-n-2}(x).$$

From these determinants, some new determinantal generators for Fibonacci and Lucas numbers may be derived, namely:

$$(4.7)\qquad \phi_n(-1) = (-1)^n(f_{n+3} - 1) \quad \text{from (3.13) and (4.2)}$$

$$(4.8)\qquad \phi_n^*(-1) = (-1)^n 2f_{n+1} \quad \text{from (3.13) and (4.3)}$$

$$(4.9)\qquad \phi_n^{**}(-1) = (-1)^n(\ell_n + 1) \quad \text{from (3.13) and (4.4)}$$

$$(4.10)\qquad \eta_n(-1) = f_n + (-1)^n \quad \text{from (3.13) and (4.6)}$$

5. Explicit Summation Expressions for Diagonal Functions

It is assumed in what follows that n is sufficiently large so that all the subscripts are greater than or equal to -1. Repeated application of the formula in (1.1) gives the lines below:

$$\begin{aligned}(5.1)\qquad r_n(x) &= 2xr_{n-1}(x) + r_{n-3}(x) \\ &= (2x)^2 r_{n-2}(x) + r_{n-3}(x) + (2x)r_{n-4}(x) \\ &= \{(2x)^3 + 1\}r_{n-3}(x) + (2x)r_{n-4}(x) + (2x)^2 r_{n-5}(x) \\ &= \{(2x)^4 + 2(2x)\}r_{n-4}(x) + (2x)^2 r_{n-5}(x) + \{(2x)^3 + 1\}r_{n-6}(x) \\ &= \{(2x)^5 + 3(2x)^2\}r_{n-5}(x) + \{(2x)^3 + 1\}r_{n-6}(x) \\ &\qquad + \{(2x)^4 + 2(2x)\}r_{n-7}(x) \\ &= \{(2x)^6 + 4(2x)^3 + 1\}r_{n-6}(x) + \{(2x)^4 + 2(2x)\}r_{n-7}(x) \\ &\qquad + \{(2x)^5 + 3(2x)^2\}r_{n-8}(x)\end{aligned}$$

One formula suggested by these lines is:

$$\begin{aligned}(5.2)\qquad r_n(x) = &\left\{\sum_{i=0}^{[j/3]} \binom{j - 2i}{i}(2x)^{j-3i}\right\} r_{n-j}(x) \\ &+ \left\{\sum_{i=0}^{[(j-2)/3]} \binom{j - 2 - 2i}{i}(2x)^{j-2-3i}\right\} r_{n-j-1}(x) \\ &+ \left\{\sum_{i=0}^{[(j-1)/3]} \binom{j - 1 - 2i}{i}(2x)^{j-1-3i}\right\} r_{n-j-2}(x)\end{aligned}$$

This may be proved by induction. Put $j + 1 = n$ in (5.2) to get

$$(5.3)\qquad r_n(x) = \sum_{i=0}^{(n-1)/3} \binom{n - 1 - 2i}{i}(2x)^{n-1-3i},$$

since $r_0(x) = r_{-1}(x) = 0$.

By substituting (5.3) in (5.2), it is found that

$$(5.4)\qquad r_n(x) = r_{j+1}(x)r_{n-j}(x) + r_{j-1}(x)r_{n-j-1}(x) + r_j(x)r_{j-n-2}(x)$$

or

$$(5.5)\qquad r_{m+n}(x) = r_{m+1}(x)r_n(x) + r_{m-1}(x)r_{n-1}(x) + r_m(x)r_{n-2}(x).$$

The identity (5.5) is similar to one found in Agronomoff [1] and Jarden [7] for third-order sequences of numbers.

Other explicit expressions for the diagonal functions include

$$(5.6)\qquad r_{-2n-1}(x) = \sum_{i=0}^{[(n-2)/3]} \binom{n-1-i}{2i+1}(-2x)^{n-2-3i}$$

$$(5.7)\qquad r_{-2n}(x) = \sum_{i=0}^{[(n-1)/3]} \binom{n-1-i}{2i}(-2x)^{n-1-3i}$$

$$(5.8)\qquad s_n(x) = (2x)^{n-1} + \sum_{i=1}^{[(n-1)/3]} \frac{n-1-i}{i}\binom{n-2-2i}{i-1}(2x)^{n-1-3i}$$

$$(5.9)\qquad s_{-2n}(x) = \sum_{i=0}^{[(n-1)/3]} \frac{n+1+i}{2i+1}\binom{n-1-i}{2i}(-2x)^{n-1-3i}$$

$$(5.10)\qquad s_{-2n-1}(x) = (-2x)^{n+1} + \sum_{i=1}^{[(n+1)/3]} \frac{n+1+i}{2i}\binom{n-i}{2i-1}(-2x)^{n+1-3i}$$

$$(5.11)\qquad t_n(x) = \sum_{i=0}^{[n/3]} \frac{n}{n-2i}\binom{n-2i}{i}(2x)^{n-3i}$$

$$(5.12)\qquad t_{-2n}(x) = \sum_{i=0}^{[n/3]} \frac{2n}{n-i}\binom{n-i}{2i}(-2x)^{n-3i}$$

$$(5.13)\qquad t_{-2n-1}(x) = \sum_{i=0}^{[(n-1)/3]} \frac{2n+1}{n-i}\binom{n-i}{2i+1}(-2x)^{n-1-3i}$$

If the method used to prove (5.3) is applied to the other sequences of Pell diagonal polynomials, then it is possible to prove that

$$(5.14)\qquad s_{m+n}(x) = r_{m+1}(x)s_n(x) + r_{m-1}(x)s_{n-1}(x) + r_m(x)s_{n-2}(x);$$

$$(5.15)\qquad t_{m+n}(x) = r_{m+1}(x)t_n(x) + r_{m-1}(x)t_{n-1}(x) + r_m(x)t_{n-2}(x).$$

The formulas (5.3) and (5.6)-(5.13) lead to some new explicit expressions for the Fibonacci and Lucas numbers:

$$(5.16)\qquad f_{n+2} - 1 = \sum_{i=0}^{[(n-1)/3]} (-1)^i\binom{n-1-2i}{i}2^{n-1-3i}$$

$$(5.17)\qquad f_{2n-1} - 1 = \sum_{i=0}^{[(n-2)/3]} \binom{n-1-i}{2i+1}2^{n-2-3i}$$

$$(5.18)\qquad f_{2n} + 1 = \sum_{i=0}^{[n/3]} \binom{n-i}{2i}2^{n-3i}$$

$$(5.19)\qquad f_n = 2^{n-2} + \sum_{i=1}^{[(n-1)/3]} \frac{n-1-i}{i}\binom{n-2-2i}{i-1}(-1)^i 2^{n-2-3i}$$

$$(5.20)\quad f_{2n} = \sum_{i=0}^{[(n-1)/3]} \frac{n+1+i}{2i+1}\binom{n-1-i}{2i}2^{n-2-3i}$$

$$(5.21)\quad f_{2n+1} = 2^n + \sum_{i=1}^{[(n+1)/3]} \frac{n+1+i}{2i}\binom{n-i}{2i-1}2^{n-3i}$$

$$(5.22)\quad \ell_n + 1 = \sum_{i=0}^{[n/3]} \frac{n}{n-2i}\binom{n-2i}{i}(-1)^i 2^{n-3i}$$

$$(5.23)\quad \ell_{2n+1} - 1 = \sum_{i=0}^{[(n-1)/3]} \frac{2n+1}{n-i}\binom{n-i}{2i+1}2^{n-1-3i}$$

$$(5.24)\quad \ell_{2n} + 1 = \sum_{i=0}^{[n/3]} \frac{2n}{n-i}\binom{n-i}{2i}2^{n-3i}$$

By Descartes' Rule, $r_n(x)$ can have no positive roots and, at most, $[(n-1)/3]$ negative roots. It is believed that this maximal number of roots is, in fact, the actual number of roots. We shall attempt to prove this in some future paper.

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