DIVISIBILITY PROPERTIES OF THE FIBONACCI NUMBERS MINUS ONE, GENERALIZED TO $C_n = C_{n-1} + C_{n-2} + k$

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1. Introduction

The numbers $\{C_n(a, b, k)\}$, defined by

$$C_n(a, b, k) = C_{n-1}(a, b, k) + C_{n-2}(a, b, k) + k,$$

with $C_1(a, b, k) = a$, $C_2(a, b, k) = b$, where k is a constant, have been studied in [1]. The Fibonacci sequence arises as the special case $F_n = C_n(1, 1, 0)$, while the Lucas sequence is $L_n = C_n(1, 3, 0)$. The sequence

$$\{C_n\} = \{\ldots, 0, 0, 1, 2, 4, 7, 12, 20, \ldots\},$$

where $C_n = C_n(0, 0, 1)$, has the property that $C_n = F_n - 1$, the sequence of Fibonacci numbers minus one.

The sequence $\{C_n\}$ has remarkable divisibility properties since almost every term is a composite number and at least one factor can always be named by examining the subscript of C_n. Further, $\{C_n\}$ contains exactly two prime terms, and two-thirds of its terms are even numbers. Analogous properties extend to the generalized sequence $\{C_n(a, b, k)\}$.

2. Prime Factors of C_n

First, since F_{3m} gives all the even Fibonacci numbers, C_{3m} is always odd, and $C_{3m\pm1}$ is always even, so the probability of choosing an even term from $\{C_n\}$ at random is 2/3. Since $C_n = F_n - 1$, we can use [2] to prove some theorems in one step.

Theorem 1: For primes of the form $p = 5k \pm 2$, p divides both C_{p-1} and C_{2p+1}.

Proof: We have $F_p \equiv -1 \pmod p$ and $F_{p+1} \equiv 0 \pmod p$ from [2]. Then

$$C_{p-1} = F_{p-1} - 1 = F_{p+1} - (F_p + 1)$$

while

$$C_{2p+1} = F_{2p+1} - 1 = (F_{p+1})^2 + (F_p + 1)F_p - (F_p + 1),$$

where all terms on the right-hand side are divisible by p in both cases.

Theorem 2: For primes of the form $p = 5k \pm 1$, p divides C_p, C_{p+1}, C_{p-2}, C_{2p-1}, C_{2p}, and C_{2p-3}.

Proof: We have $F_p \equiv 1 \pmod p$ and $F_{p-1} \equiv 0 \pmod p$ from [2]. We write C_p, C_{p+1}, and C_{p-2} in forms in which p divides the terms on the right-hand side:

$$C_p = (F_p - 1),$$

$$C_{p+1} = F_{p+1} - 1 = F_{p-1} + (F_p - 1),$$

$$C_{p-2} = F_{p-2} - 1 = (F_p - 1) - F_{p-1}.$$

Since

$$C_{p+n-1} = F_{p+n-1} - 1 = F_p(F_n - 1) + F_{p-1}F_{n-1} + (F_p - 1),$$

where $p|F_{p-1}F_{n-1}$ and $p|(F_p - 1)$ but p does not divide F_p, observe that whenever $p|(F_n - 1)$, then $p|C_{p+n-1}$. Let $n = p$, $p + 1$, and $p - 2$ to write that

$$p|C_{2p-1},\ p|C_{2p},\ \text{and}\ p|C_{2p-3}.$$

Further, a little rewriting lets us prove the following corollary.

Corollary: If $p|C_n$, then $p|C_{n+m(p-1)}$, $m = 0, \pm 1, \pm 2, \ldots$, where p is a prime of the form $5k \pm 1$.

Proof: From the proof of Theorem 2, if $p|C_n$, then $p|C_{n+(p-1)}$. The corollary holds by the Axiom of Mathematical Induction, since whenever $p|C_{n+m(p-1)}$, then

$$p|C_{[n+m(p-1)]+(p-1)} = C_{n+(m+1)(p-1)}.$$

Theorem 3: If $\Pi(p)$ is the period of a prime p in the Fibonacci sequence modulo p, then

$$p|C_{k\Pi(p)-1},\quad p|C_{k\Pi(p)+1},\quad \text{and}\ p|C_{k\Pi(p)+2}.$$

Proof: Since

$$C_{k\Pi(p)+n} - C_n = F_{k\Pi(p)+n} - F_n,$$

and since p divides the right-hand side by definition of $\Pi(p)$, if $p|C_n$, then $p|C_{k\Pi(p)+n}$. Theorem 3 follows because $C_{-1} = C_1 = C_2 = 0$.

Corollary: The prime 5 divides C_{20k-1}, C_{20k+1}, C_{20k+2}, and C_{20k+8}.

Proof: $\Pi(5) = 20$, and 5 divides C_{-1}, C_1, C_2, and C_8.

Theorem 4: If p is a prime of the form $5k \pm 2$, then $p|C_{q(p+1)-2}$ if q is odd. If q is even, $p|C_{q(p+1)-1}$, $p|C_{q(p+1)+1}$, and $p|C_{q(p+1)+2}$.

Proof: If $p|C_n$, then $p|C_{n+m\Pi(p)}$ as in the proof of Theorem 3. From [3], if p is a prime of the form $5k \pm 2$, then $\Pi(p)|2(p + 1)$. Then, $p|C_{n+2m(p+1)}$, m any integer. Since

$$p|C_{p-1},\ p|C_{p-1+2m(p+1)} = C_{(2m+1)p+(2m-1)},$$

or, for q odd,

$$p|C_{qp+(q-2)} = C_{q(p+1)-2}.$$

If q is even, let $q(p + 1) = k\Pi(p)$ for some k, since $\Pi(p)|2(p + 1)$, and use Theorem 3.

Corollary: If $p = 5k \pm 2$, then

(i) p divides $C_{(p+2)(p-1)}$, $C_{p(p+3)}$, and $C_{p^s(p+1)-2}$;

(ii) p divides $C_{p(p+2)}$, C_{p^2-2}, C_{p^2}, and C_{p^2+1}.

Proof: (i) Take q odd, $q = p$, $q = p + 2$, and $q = p^s$, in Theorem 4. To show (ii), take q even, $q = p + 1$, $q = p - 1$.

Theorem 5: If p is a prime of the form $5k \pm 1$, then

$$p|C_{(m+1)p-(m+2)},\quad p|C_{(m+1)p-(m-1)},\quad \text{and}\ p|C_{(m+1)p-m}\ \text{for any integer}\ m.$$

Proof: From the Corollary to Theorem 2, if $p|C_n$, then $p|C_{n+m(p-1)}$. From Theorem 2, take $n = p - 2$, $p + 1$, and $n = p$, and simplify.

Corollary: For any prime p, $p \neq 5$, $p|C_{p^2}$, $p|C_{p^2+1}$, and $p|C_{p^2-2}$.

Proof: If $p = 5k \pm 1$, let $m = p$ in Theorem 5. If $p = 5k \pm 2$, use the Corollary to Theorem 4.

Theorem 6: If $\Pi(j)$ is the period of any integer j, $j \neq 0$, in the Fibonacci sequence modulo j, then, for all integers k,

$$j|C_{k\Pi(j)-1}, \quad j|C_{k\Pi(j)+1}, \quad \text{and } j|C_{k\Pi(j)+2}.$$

Proof: See the proof of Theorem 3. Notice that any integer will eventually divide C_n for some n.

3. Fibonacci and Lucas Factors of C_n

Since $C_{m+n} - C_{m-n} = F_{m+n} - F_{m-n}$, we can write

$$(3.1) \qquad C_{m+n} - C_{m-n} = F_m L_n, \quad \text{if } n \text{ is odd,}$$
$$C_{m+n} - C_{m-n} = L_m F_n, \quad \text{if } n \text{ is even.}$$

Observe that, if $L_n|C_{m-n}$, then $L_n|C_{m+n}$, and L_n has period $2n$ if n is odd. Similarly, F_n has period $2n$ if n is even. Putting these together with Theorem 6, we write

Theorem 7: If n is odd, L_n divides C_{2rn-1}, C_{2rn+1}, and C_{2rn+2}, while if n is even, F_n divides C_{2rn-1}, C_{2rn+1}, and C_{2rn+2} for any integer r.

Now things are getting exciting. Since we can take $n = 2k + 1$ to find that L_{2k+1} divides C_{4k+1}, C_{4k+3}, and C_{4k+4}, and $n = 2k$ to see that F_{2k} divides C_{4k-1}, C_{4k+1}, and C_{4k+2}, notice that C_n is always divisible either by L_{2k+1} or by F_{2k}. Now, if $k = 1$, $F_2 = 1$ divides any integer, so take $|k| \geq 2$. Thus, if $n \geq 7$, or if $n \leq -5$, then C_n always has at least one factor smaller than C_n and greater than 1 which we can write exactly, so C_n is not prime. We examine the sequence from C_{-4} through C_6: -4, 1, -2, 0, -1, 0, 0, 1, 2, 4, 7, and find that the only primes are 2 and 7.

Theorem 8: The sequence of Fibonacci numbers minus one, $C_n = F_n - 1$, contains only composite numbers for all $n \geq 7$ and all $n \leq -5$. The only primes which appear in $\{C_n\}$ are $C_4 = 2$, $C_6 = 7$, and $|C_{-2}| = 2$.

4. Divisibility of the Generalized Sequence $\{C_n(a, b, k)\}$

From [1], the sequence $\{C_n(a, b, k)\}$ with initial values $C_1 = a$ and $C_2 = b$ is given by

$$(4.1) \qquad C_n(a, b, k) = C_{n-1}(a, b, k) + C_{n-2}(a, b, k) + k$$
$$= aF_{n-2} + bF_{n-1} + kC_n(0, 0, 1)$$
$$= H_n + kC_n$$

for the generalized Fibonacci numbers H_n, $H_n = C_n(a, b, 0)$, and $C_n(0, 0, 1) = C_n$ of the earlier section.

As in Section 3,

$$C_{m+n}(a, b, k) - C_{m-n}(a, b, k) = (H_{m+n} - H_{m-n}) + k(C_{m+n} - C_{m-n}),$$

so that we can write

$$(4.2) \quad C_{m+n}(a, b, k) - C_{m-n}(a, b, k) = L_n H_m + kF_m L_n, \text{ if } n \text{ is odd;}$$
$$C_{m+n}(a, b, k) - C_{m-n}(a, b, k) = F_n(H_{m+1} + H_{m-1}) + kL_m F_n, \text{ if } n \text{ is even.}$$

Thus, the periods of F_n and L_n are still $2n$, where we again distinguish n even and n odd. Also, since every nonzero integer eventually divides F_k for some k, every nonzero integer will divide $C_n(a, b, k)$ for some n if $\{C_n(a, b, k)\}$ contains a zero term. If $\{C_n(a, b, k)\}$ contains two zero terms, in some cases we will again have a finite number of primes occurring.

Theorem 9: If $C_q(a, b, k) = 0$, and if a nonzero integer j has period $\Pi(j)$ in the Fibonacci sequence, then $j | C_{q+m\Pi(j)}(a, b, k)$ for all integers m.

Theorem 10: If $F_{2m} | C_q(a, b, k)$, then

$$F_{2m} | C_{q+4m}(a, b, k),$$

and if $L_{2m+1} | C_q(a, b, k)$, then

$$L_{2m+1} | C_{q+4m+2}(a, b, k),$$

for any integer m.

Now, Theorem 10 gives us some interesting special cases. Notice that if $C_q(a, b, k) = 0$, and if $C_{q+r}(a, b, k) = 0$, where r is an odd number, then $\{C_n(a, b, k)\}$ will contain a finite number of primes, because for n larger than certain beginning values, $C_n(a, b, k)$ will always be divisible either by F_{2m} or L_{2m+1}, where $F_{2m} \neq 0, \pm 1$, and $L_{2m+1} \neq \pm 1$.

Without loss of generality, if $\{C_n(a, b, k)\}$ has a zero term, renumber the terms, taking new starting values, so that

$$a = 0 = C_1(0, b, k).$$

Then, if $C_{r+1}(0, b, k) = 0$ for some $r > 0$, from (4.1),

$$C_{r+1}(0, b, k) = 0 \cdot F_{r-1} + bF_r + kC_{r+1} = 0,$$

where we list some possibilities and special cases. Notice that $k = F_r$ and $b = -C_{r+1} = -F_{r+1} + 1$ always is a solution, and write the resulting

$$C_n(a, b, k) = C_n(0, -C_{r+1}, F_r).$$

For $r = 1$, we have $C_n(0, 0, 1) = C_n$; for $r = 2$, $C_n(0, -1, 1) = C_{n-2}$; and $r = 3$ gives $C_n(0, -2, 2) = 2C_{n-2}$, all the sequence of Fibonacci numbers minus one.

Consider $r = 4$ and $\{C_n(0, -4, 3)\} = \{\ldots, 0, -4, 1, -2, 0, 1, 4, 8, 15, 26, 44, 73, 120, \ldots\}$. We can show that

$$C_n(0, -4, 3) = -4F_{n-1} + 3C_n = L_{n-3} - 3.$$

From [2], we have $L_{2p} \equiv 3 \pmod p$ where p is any prime, so $p | L_{2p} - 3$, and we have

$$p | C_{2p+3}(0, -4, 3).$$

All odd-subscripted $C_n(0, -4, 3)$ have F_m or L_m for a divisor for some m, but we cannot easily say whether or not $\{C_n(0, -4, 3)\}$ contains a finite number of primes. However, any prime terms will have a subscript of the form $6m$. If r is even, we cannot determine whether or not $\{C_n(0, b, k)\}$ will contain a finite number of prime terms.

However, for $r = 5$, $\{C_n(0, -7, 5)\}$ contains only two primes, 2 and 7. We write $C_n(0, -7, 5)$ for $-3 \le n \le 10$: $-24, 7, -12, 0, -7, -2, -4, -1, 0, 4, 9, 18, 32$. We observe $|C_1| = 2$ and $|C_3| = 7 = C_{-2}$. From Theorem 10,

$$L_{2k+1} | C_{1+4k+2},\ L_{2k+1} | C_{6+4k+2},\ F_{2k} | C_{1+4k}, \text{ and } F_{2k} | C_{6+4k},$$

covering every possible subscript, so that $C_n(0, -7, 5)$ always has F_{2k} or L_{2k+1} for a divisor. But $F_{2k} = \pm 1$ for $k = \pm 1$, and $L_{2k+1} = \pm 1$ for $k = 0$ and $k = -1$. So terms $C_n(0, -7, 5)$ for $n > 10$ or $n < -3$ have a divisor greater than 1 and less than $C_n(0, -7, 5)$ and thus are not prime. For $r = 7$, in a similar fashion, we find only the three primes 7, 73, and 79 in $\{C_n(0, -20, 13)\}$. If $r = 9$, all the terms of $\{C_n(0, -54, 34)\}$ are even, but, if we instead consider $\{C_n(0, -27, 17)\}$, we find

$$|C_5| = 13 = C_{11},\ |C_8| = 11, \text{ and } C_{14} = 107$$

as the only primes. Finally, $r = 11$ has only two primes

$$|C_5| = 73 \text{ and } |C_8| = 79,$$

but $r = 13$ is the best of all, containing no primes at all!

From the preceding discussion, we can write the following theorem.

Theorem 11: If $\{C_n(a, b, k)\}$ has $C_1(a, b, k) = 0$ and $C_{1+r}(a, b, k) = 0$ for r an odd integer, then $|C_n(a, b, k)|$ is prime for only a finite number of values for n.

Now, recall from above that the probability of choosing an even term from $\{C_n\} = \{C_n(0, 0, 1)\}$ is 2/3. $\{C_n(a, b, k)\}$ has the same property only when k is odd, and when at least one of a or b is even. These results can be verified by examining $C_n(a, b, k)$ from (4.1) for $n = 3m$, $3m + 1$, and $3m + 2$, where we always take k odd.

(i) $C_{3m}(a, b, k) = aF_{3m-2} + bF_{3m-1} + kC_{3m}$.

Note that kC_{3m}, F_{3m-1}, and F_{3m-2} are all odd. Then, if a and b have the same parity, $C_{3m}(a, b, k)$ is odd, while if a and b have opposite parity, $C_{3m}(a, b, k)$ is even.

(ii) $C_{3m+1}(a, b, k) = aF_{3m-1} + bF_{3m} + kC_{3m+1}$.

Here both bF_{3m} and kC_{3m+1} are always even, while F_{m-1} is odd, so $C_{3m+1}(a, b, k)$ is even or odd as a is even or odd.

(iii) $C_{3m+2}(a, b, k) = aF_{3m} + bF_{3m+1} + kC_{3m+2}$.

Now, aF_{3m} and kC_{3m+2} are always even, while F_{3m+1} is odd, so $C_{3m+2}(a, b, k)$ is even or odd as b is even or odd.

Putting the three cases together, first notice that, if all of a, b, and k are odd, $C_n(a, b, k)$ is always odd. If a and b are both even, $C_{3m}(a, b, k)$ is odd but $C_{3m+1}(a, b, k)$ and $C_{3m+2}(a, b, k)$ are both even. If a and b have opposite parity, $C_{3m}(a, b, k)$ is even, and either $C_{3m+1}(a, b, k)$ or $C_{3m+2}(a, b, k)$ is even, but not both. Then, if k is odd, and at least one of a or b is even, the probability that a term chosen at random from $\{C_n(a, b, k)\}$ will be even is 2/3.

Next, re-examine the three cases for k even. If a, b, and k are all even, $C_n(a, b, k)$ is always even, a trivial result. In (i), kC_{3m} is even, while F_{3m-2} and F_{3m-1} are odd, so that $C_{3m}(a, b, k)$ is odd if a and b have opposite parity, but even if a and b have the same parity. From (ii), both bF_{3m} and kC_{3m+1} are even, while F_{3m-1} is odd, so $C_{3m+1}(a, b, k)$ is even or odd as a is even or odd. From (iii), both aF_{3m} and kC_{3m+2} are even, while F_{3m+1} is odd, so $C_{3m+2}(a, b, k)$ is even or odd as b is even or odd. Putting these results together, if k is even, and a and b have opposite parity, then $C_{3m}(a, b, k)$ is odd while exactly one of $C_{3m+1}(a, b, k)$ or $C_{3m+2}(a, b, k)$ is odd. If k is even and both a and b are odd, $C_{3m}(a, b, k)$ is even but both $C_{3m+1}(a, b, k)$ and $C_{3m+2}(a, b, k)$ are odd. Thus, if k is even and at least one of a or b is odd, the probability of randomly choosing an even term from $\{C_n(a, b, k)\}$ is 1/3. We summarize in Theorem 12.

Theorem 12: If k is odd, and at least one of a or b is even, the probability that a term chosen at random from $\{C_n(a, b, k)\}$ will be even is 2/3. If k is even, and at least one of a or b is odd, the probability that a term chosen at random from $\{C_n(a, b, k)\}$ will be even is 1/3. If a, b, and k are all odd, $C_n(a, b, k)$ is always odd.

References

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