SERIES TRANSFORMATIONS FOR FINDING RECURRENCES FOR SEQUENCES

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There is no *really* new theoretical result below. However, our paper will show how to use an old and clever idea in order to discover recurrences. Such an expository paper surveying these techniques may be of interest. A few specific books or papers are needed, but for general background as to notations and definitions for Fibonacci, Bernoulli, Bell, and Stirling numbers, etc., the reader may consult papers in the *Fibonacci Quarterly* or Riordan's books [6], [7]. Niven [5] has given a good, readable account of formal power series. It is shown there when and why convergence questions may be ignored. Finally, four papers of the author, [1], [2], [3], and [4], may be consulted for other background information. Reference [1] is especially useful for an abundance of intricate generating functions for powers of Fibonacci numbers.

We begin with a small theorem about *formal power series*.

Theorem 1. Exponential Series Transformation: Define

$$S(n) = \sum_{k=0}^{n} \binom{n}{k} A_k, \tag{1}$$

$$\mathscr{A}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} A_n, \tag{2}$$

and

$$\mathscr{G}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} S(n). \tag{3}$$

Then

$$\mathscr{G}(x) = e^x \mathscr{A}(x). \tag{4}$$

The proof is simple and runs as follows. We have

$$\begin{aligned}\mathscr{G}(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} \binom{n}{k} A_k = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \frac{A_k}{(n-k)!k!} \\ &= \sum_{k=0}^{\infty} \frac{A_k}{k!} \sum_{n=k}^{\infty} \frac{x^n}{(n-k)!} = \sum_{k=0}^{\infty} \frac{A_k}{k!} \sum_{n=0}^{\infty} \frac{x^{n+k}}{n!} \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} A_k \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \mathscr{A}(x).\end{aligned}$$

What we wish to show here is that by clever manipulation, especially if e^x combines in a *novel* way with $\mathscr{A}$, we may often use (4) to find a *different* way of writing expansion (3) that does not use $S(n)$ again directly. Then, by equating coefficients, we get a *new* recurrence. This is a common piece of psychological trickery used in research. We say the same thing but in a seemingly different manner.

Relation (1) may easily be inverted to give

$$A_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} S(k), \tag{5}$$

which is a well-known result [7] which follows readily from the Kronecker delta

$$(6)\qquad \sum_{k=j}^{n}(-1)^{n-k}\binom{n}{k}\binom{k}{j} = \binom{0}{n-j} = \begin{cases}1 & \text{if } j = n,\\ 0 & \text{if } j \neq n.\end{cases}$$

As a consequence of this inversion, we may also state Theorem 1 in a dual form.

Theorem 1': Define

$$(1')\qquad A_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}S(k),$$

$$(2')\qquad \mathscr{G}(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}S(n),$$

$$(3')\qquad \mathscr{A}(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}A_n.$$

Then

$$(4')\qquad \mathscr{A}(x) = e^{-x}\,\mathscr{G}(x).$$

We will now concentrate on applications of Theorem 1.

Application 1. Let $A_n = (-1)^n F_n$, where F_n is the n^{th} Fibonacci number defined by

$$F_{n+1} = F_n + F_{n-1},\ F_0 = 0,\ F_1 = 1.$$

We must recall that the exponential generating function for the Fibonacci numbers is

$$(7)\qquad \sum_{n=0}^{\infty}\frac{x^n}{n!}F_n = \frac{e^{ax} - e^{bx}}{a - b},$$

where $a + b = 1$, $ab = -1$. These are the roots of the characteristic equation associated with the recurrence relation. In fact, $a, b = (1 \pm \sqrt{5})/2$.

It then follows in this special Fibonacci case that

$$\mathscr{G}(x) = -\mathscr{A}(-x).$$

To show this, we have

$$\mathscr{G}(x) = e^x\mathscr{A}(x) = e^x\frac{e^{-ax} - e^{-bx}}{a - b} = \frac{e^{(1-a)x} - e^{(1-b)x}}{a - b} = \frac{e^{bx} - e^{ax}}{a - b}$$

$$= -\frac{e^{ax} - e^{bx}}{a - b} = -\mathscr{A}(-x) = -\sum_{n=0}^{\infty}\frac{x^n}{n!}F_n.$$

Recalling (1) and (3), we have, upon equating coefficients, the new recurrence relation $S(n) = -F_n$, i.e.,

$$(8)\qquad \sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}F_k = F_n.$$

The reader may find it interesting to try to provide a *simple inductive proof* of relation (8) using the binomial and Fibonacci recurrences

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1},\ F_{n+1} = F_n + F_{n-1}.$$

Such a proof requires a certain algebraic skill.

Application 2. Let $A_n = B_n$, the n^{th} Bernoulli number, whose exponential generating function is known to be

$$(9)\qquad \mathscr{A}(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}B_n = \frac{x}{e^x - 1}.$$

It can then easily be seen that

$$\mathscr{G}(x) = e^x \frac{x}{e^x - 1} = \frac{-x}{e^{-x} - 1} = \mathscr{A}(-x),$$

and it thus follows from Theorem 1 that $S(n) = (-1)^n B_n$, i.e.,

$$(10) \qquad \sum_{k=0}^{n} \binom{n}{k} B_k = (-1)^n B_n, \text{ valid for all } n \geq 0.$$

Remark: Because $B_n = 0$ for all odd $n \geq 3$, this familiar recurrence may be modified to read as

$$(11) \qquad \sum_{k=0}^{n} \binom{n}{k} B_k = B_n, \text{ valid for all } n \geq 2.$$

Symbolically, in the umbral notation of Blissard, this is often written in the compact form $(B + 1)^n = B^n$ (expand and demote powers to subscripts).

Application 3. Let $A_n = B(n)$, the n^{th} Bell, or exponential number. These numbers have the well-known exponential generating function

$$(12) \qquad e^{e^x - 1} = \exp(e^x - 1) = \sum_{n=0}^{\infty} \frac{x^n}{n!} B(n),$$

so this is our $\mathscr{A}(x)$.

By out theorem, using relation (4), we find that

$$\mathscr{G}(x) = e^x \exp(e^x - 1) = D_x \exp(e^x - 1) = D_x \mathscr{A}(x),$$

$$= \sum_{n=0}^{\infty} \frac{nx^{n-1}}{n!} B(n) = \sum_{n=0}^{\infty} \frac{x^n}{n!} B(n + 1),$$

whence by our theorem we find the recurrence relation $S(n) = B(n + 1)$, i.e.,

$$(13) \qquad \sum_{k=0}^{n} \binom{n}{k} B(k) = B(n + 1), \text{ valid for all } n \geq 0.$$

By the inversion (5), this yields

$$(14) \qquad \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} B(k + 1) = B(n),$$

which, in terms of the finite difference quotient operator, says that

$$(15) \qquad \underset{k,1}{\Delta^n} B(k + 1) = B(n),$$

which is the analogue of the differential calculus formula

$$(16) \qquad (D_x)^n e^x = e^x.$$

This parallel of (15) with (16) is a further reason why the Bell numbers are reasonably called "exponential" numbers.

The reader may look for other examples where a generating function has some nice relation to the exponential function, which is part of the secret of success. Such research requires an artistic touch of intuition.

It is possible to set down a parallel theorem for binomial generating functions. We offer the following.

Theorem 2. Binomial Series Transformation: Define as before in (1),

$$(17) \qquad S(n) = \sum_{k=0}^{n} \binom{n}{k} A_k,$$

(18) $$\mathscr{B}(x) = \sum_{n=0}^{\infty} x^n A_n,$$

and

(19) $$\mathscr{H}(x) = \sum_{n=0}^{\infty} x^n S(n).$$

Then

(20) $$\mathscr{H}(x) = \sum_{n=0}^{\infty} A_n \frac{x^n}{(1-x)^{n+1}}.$$

and the best we can do to parallel (4) is to write this as

(21) $$\mathscr{H}(x) = \frac{1}{1-x}\mathscr{B}(z), \text{ where } z = \frac{x}{1-x}.$$

The proof is easy and runs as follows. We have

$$\begin{aligned}\mathscr{H}(x) &= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{n}{k} A_k = \sum_{k=0}^{\infty} A_k \sum_{n=k}^{\infty} \binom{n}{k} x^n \\ &= \sum_{k=0}^{\infty} A_k \sum_{n=0}^{\infty} \binom{n+k}{k} x^{n+k} = \sum_{k=0}^{\infty} x^k A_k \sum_{n=0}^{\infty} \binom{n+k}{k} x^n \\ &= \sum_{k=0}^{\infty} x^k A_k (1-x)^{-k-1} = \frac{1}{1-x}\sum_{k=0}^{\infty} A_k z^k = \frac{1}{1-x}\mathscr{B}(z).\end{aligned}$$

This result is useful in a different way than Theorem 1. We give as an example,

Application 4. Let $A_n = (-1)^n F_n$ as in Application 1. Then

$$\mathscr{B}(x) = \sum_{n=0}^{\infty} (-x)^n F_n = \frac{-x}{1+x-x^2}$$

and

$$\begin{aligned}\mathscr{H}(x) &= \frac{1}{1-x}\mathscr{B}(z) = \frac{1}{1-x}\,\frac{-z}{1+z-z^2} = \frac{-x}{1-x-x^2} \\ &= -\mathscr{B}(-x) = -\sum_{n=0}^{\infty} F_n x^n,\end{aligned}$$

so that by Theorem 2 we have the recurrence $S(n) = -F_n$, i.e.,

(22) $$\sum_{k=0}^{n} (-1)^{k+1}\binom{n}{k} F_k = F_n,$$

which is precisely result (8) again, but it required a bit more work to obtain it by use of Theorem 2. This gives some feeling for the elegance of the exponential generating function when it can be used.

Application 5. In Theorem 2, let $A_n = F_n$ using the Fibonacci numbers again. Then

$$\mathscr{B}(x) = \frac{x}{1-x-x^2}$$

and the reader may verify that a bit of algebra using $A_n = 1$ and $m = 2$ in equation (2.11) in [1] yields

(23) $$\mathscr{H}(x) = \frac{x}{1-3x+x^2} = \sum_{n=0}^{\infty} F_{2n} x^n,$$

so that we have the recurrence $S(n) = F_{2n}$, i.e.,

(24) $$\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n}.$$

Application 6. Let us apply Theorem 1 to a generating function studied by Euler (cf. [2], p. 48, and [4], Sect. 6). Euler used the generating function

(25) $\mathscr{A}(x) = \mathscr{A}(x, p) = (e^x - 1)^p$

to evaluate the series

(26) $S(n, p) = \dfrac{1}{p!}\displaystyle\sum_{j=0}^{p}(-1)^{p-j}\binom{p}{j}j^n,$

which we have designated here by the "Stirling Number of Second Kind" notation of Riordan. It is known (see [4], Sect. 6) that

(27) $\mathscr{A}(x, p) = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{k=0}^{p}(-1)^k\binom{p}{k}k^n.$

In Theorem 1 then, with this for $\mathscr{A}(x)$, and taking $S(n)$ to be given by

(28) $S(n) = \displaystyle\sum_{i=0}^{n}\binom{n}{i}\sum_{k=0}^{p}(-1)^{p-k}\binom{p}{k}k^i,$

(29) $A_n(p) = \displaystyle\sum_{k=0}^{p}(-1)^{p-k}\binom{p}{k}k^n,$

we then find by Theorem 1 that

$$\begin{aligned}\mathscr{G}(x) &= e^x\mathscr{A}(x, p) = e^x(e^x - 1)^p = (e^x - 1 + 1)(e^x - 1)^p\\ &= (e^x - 1)(e^x - 1)^p + (e^x - 1)^p = (e^x - 1)^{p+1} + (e^x - 1)^p\end{aligned}$$

or, more simply,

(30) $\mathscr{G}(x) = \mathscr{A}(x, p + 1) + \mathscr{A}(x, p).$

Therefore,

(31) $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}S(n) = \sum_{n=0}^{\infty}\frac{x^n}{n!}[A_n(p + 1) + A_n(p)],$

so that we find the recurrence

(32) $S(n) = A_n(p + 1) + A_n(p),$

which, in view of (28) and (29), says

(33) $\displaystyle\sum_{j=0}^{n}\binom{n}{j}A_j(p) = A_n(p + 1) + A_n(p).$

Comparing (26) and (29), we have the correspondence

(34) $A_n(p) = p!S(n, p)$

for translating our results into Riordan's "Stirling Number" notation. Thus, we find

(35) $\displaystyle\sum_{k=0}^{n}\binom{n}{k}S(k, p) = (p + 1)S(n, p + 1) + S(n, p)$

or

$$\sum_{n=0}^{n-1}\binom{n}{k}S(k, p) = (p + 1)S(n, p + 1).$$

However, $S(k, p) = 0$ whenever $0 \le j < p$, so we finally get the recurrence formula for the Stirling Numbers of the Second Kind, i.e.,

(36) $\displaystyle\sum_{k=p}^{n-1}\binom{n}{k}S(k, p) = (p + 1)S(n, p + 1).$

Conclusion. The work we have presented here was based on the use of the binomial coefficient $\binom{n}{k}$ in the defining relationships (1) and (17). It is easy to replace this by other functions $g(n, k)$ and obtain parallel theorems. We just have to impose interesting properties on $g(n, k)$ in order to get interesting theorems. In later papers we will exhibit such results for q-analogs, Fibonomial coefficients, and the bracket function.

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References

1. H. W. Gould. "Generating Functions for Products of Powers of Fibonacci Numbers." *Fibonacci Quarterly 1* (1963), No. 2, 1-16.
2. H. W. Gould. "Explicit Formulas for Bernoulli Numbers." *Amer. Math. Monthly 79* (1972):44-51.
3. H. W. Gould. "Coefficient Identities for Powers of Taylor and Dirichlet Series." *Amer. Math. Monthly 81* (1974):3-14.
4. H. W. Gould. "Euler's Formula for n^{th} Differences of Powers." *Amer. Math. Monthly 85* (1978):450-67.
5. Ivan Niven. "Formal Power Series." *Amer. Math. Monthly 76* (1969):871-89.
6. John Riordan. *An Introduction to Combinatorial Analysis*. New York: John Wiley & Sons, 1958.
7. John Riordan. *Combinatorial Identities*. New York: John Wiley & Sons, 1968.

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