A HYPERCUBE PROBLEM

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1. Introduction

The n-dimensional hypercube, Q_n, is the graph whose vertex set, $V(Q_n)$, is the set of all n-bit strings, any two of which are adjacent iff they differ in exactly one bit. We refer to Q_n as the n-cube. The 1-, 2-, 3-, and 4-cubes are illustrated in Figure 1.

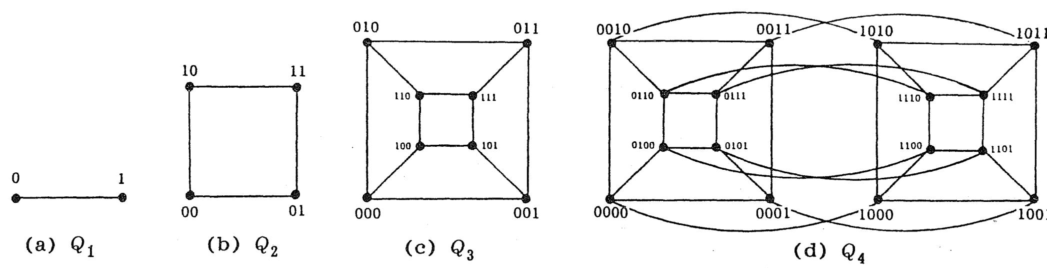


FIGURE 1

Sometime in the early 1980s, Paul Erdös asked for the largest order of an induced subgraph of Q_n which contains no 4-cycle. This question has been answered and extremal graphs characterized [1]. Since a 4-cycle in Q_n can be interpreted as a sub-Q_2, it is natural to generalize and ask for the order of a largest induced subgraph of Q_n which contains no sub-Q_k, $k \in \{1, 2, 3, \ldots\}$. It is also natural to ask for the order of a largest induced subgraph of Q_n which contains no $2k$-cycle, $k \in \{2, 3, \ldots\}$, but this question seems far more difficult. Partial results in this direction appear in [2].

With the advent of the hypercube computer, these questions assume a new significance. An n-dimensional hypercube computer is a multicomputer with 2^n processors, possessing the network topology of an n-dimensional hypercube; i.e., each vertex of the cube is associated with a processor and each edge represents a direct communication link between the two processors incident with that edge. A question that has generated some interest recently ([3], [4]) is *how does the hypercube computer behave in the presence of faulty nodes (or links)*? In particular, given a set of faulty nodes (links), what is the largest subcube that remains? The question is pertinent because there are algorithms which are designed to run on a cube structure, and in the presence of faulty nodes (links) will run on the largest remaining subcube [3].

In the following, F_n and L_n will denote the n^{th} Fibonacci and Lucas numbers, respectively, having the initial conditions $F_0 = 0$, $F_1 = 1$ and $L_1 = 1$, $L_2 = 3$. We use $\lfloor x \rfloor$ and $\lceil x \rceil$ to denote the greatest integer less than or equal to x and the least integer greater than or equal to x, respectively. Now, let $f(n, k)$ denote the largest order of an induced subgraph of Q_n that contains no sub-Q_k. It is known that

$$f(n,\ 2) = \left\lceil \frac{2}{3} \cdot 2^n \right\rceil \quad [1].$$

A good lower bound for $f(n,\ 3)$ is known, namely,

$$f(n,\ 3) \geq \frac{3}{4} \cdot 2^n + 2^{\left\lfloor \frac{n-2}{2} \right\rfloor} \quad [5].$$

In general, it is easy to show [3] that

$$(1) \qquad f(n,\ k) \geq \frac{k}{k+1} \cdot 2^n.$$

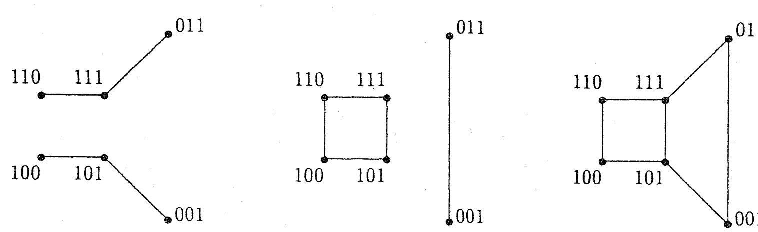
In this paper we prove, in Theorem 2 and its corollary, a result which enables us to improve on the inequality in (1) for the special case $k = 4$. We obtain

$$(2) \qquad f(n,\ 4) \geq \begin{cases} \frac{4}{5} \cdot 2^n + \frac{1}{5} L_{n+1}, & n \text{ even}, \\ \frac{4}{5} \cdot 2^n + \frac{2}{5} L_n, & n \text{ odd}. \end{cases}$$

2. The Hypercube Problem

The *order* of a graph is the size of its vertex set. Given a graph G with vertex set $V(G)$ and edge set $E(G)$, a *subgraph of* G is a graph whose vertex and edge sets are subsets of $V(G)$ and $E(G)$, respectively. If H is a subgraph of Q_n and there is a subgraph of H isomorphic to some Q_k, $1 \leq k \leq n$, then H is said to contain a sub-Q_k. Given any graph G with vertex set $V(G)$ and $S \subseteq V(G)$, the subgraph of G which is *induced* by S, denoted $\langle S \rangle$, is the graph with vertex set S and two vertices of $\langle S \rangle$ are adjacent iff they are adjacent in G.

In Figure 2, G_1, G_2, and G_3 are all subgraphs of Q_3. The graphs G_1 and G_2 are not induced subgraphs of Q_3, while G_3 is. G_2 and G_3 both contain a sub-Q_2.



(a) G_1 (b) G_2 (c) G_3

FIGURE 2

Example: Let W be the set of 16 vertices listed in Figure 3. For each $v = v_1v_2v_3v_4v_5v_6v_7$ in W, we have $v_5 = v_6 = v_7 = 1$, while the first four bits range from 0000 to 1111. Hence $\langle W \rangle$, the subgraph of Q_7 induced by W, contains a sub-Q_4. (In fact, $\langle W \rangle$ is isomorphic to Q_4.)

For $v \in V(Q_n)$, the *weight* of v, denoted wgt(v), is defined to be the number of 1's in v. Observe that the vertices of W have weights ranging from 0 to 4 (mod 5). In fact, for all n, any sub-Q_4 in Q_n contains vertices with weights of 0, 1, 2, 3, and 4 (mod 5). For $n \in Z^+$, $k \in \{0, 1, 2, 3, 4\}$, let

$$V_k^n = \{v \in V(Q_n) : \text{wgt}(v) \equiv k \pmod 5\}.$$

If $V \subseteq V(Q_n)$ and $\langle V \rangle$ contains a sub-Q_4, then

$$V \cap V_k^n \neq \emptyset \text{ for all } k \in \{0, 1, 2, 3, 4\}.$$

0	0	0	0	1	1	1
0	0	0	1	1	1	1
0	0	1	0	1	1	1
0	0	1	1	1	1	1
0	1	0	0	1	1	1
0	1	0	1	1	1	1
0	1	1	0	1	1	1
0	1	1	1	1	1	1
1	0	0	0	1	1	1
1	0	0	1	1	1	1
1	0	1	0	1	1	1
1	0	1	1	1	1	1
1	1	0	0	1	1	1
1	1	0	1	1	1	1
1	1	1	0	1	1	1
1	1	1	1	1	1	1

FIGURE 3. The vertex set W

Hence for any k, $\langle V(Q_n) - V_k^n \rangle$ contains no sub-Q_4. This implies (1). To obtain the inequality in (2), we first let $V_k^n = \# V_k^n$. Clearly,

$$V_k^n = \sum_{\substack{j \equiv k \\ \text{mod } 5}} \binom{n}{j},$$

and if we define

$$V(n) = \min_{0 \le k \le 4} V_k^n,$$

then we obtain $f(n, k) \ge 2^n - V(n)$. Determination of a formula for $V(n)$ is the content of the next two sections.

3. Properties of the V_k^n

We begin with an example. By definition,

$$V_0^7 = \binom{7}{0} + \binom{7}{5} = 1 + 21 = 22 = V_2^7 = \binom{7}{2} + \binom{7}{7}.$$

Similarly,

$$V_1^7 = \binom{7}{1} + \binom{7}{6} = 14 \quad \text{and} \quad V_3^7 = \binom{7}{3} = \binom{7}{4} = V_4^7 = 35.$$

Hence, $V(7) = 14 = V_1^7$. On the other hand, if we compute values of V_k^6, we find that $V(6) = V_0^6 = V_1^6$. In Theorem 1 we will show that, if we define

$$k(n) = \left\lfloor \frac{n}{2} \right\rfloor - 2 \pmod 5,$$

then $V(n) = V_{k(n)}^n$.

Because the terms V_k^n are computed in terms of binomial coefficients, we would expect the V_k^n to reflect some of the properties of binomial coefficients. That this is the case is illustrated in the following lemma.

Lemma: For $n \in Z^+$, $k \in \{0, 1, 2, 3, 4\}$,

(1) *(Recursion Formula)*

$$V_k^n = V_k^{n-1} + V_{k-1}^{n-1}, \text{ where } k - 1 \text{ is computed modulo 5.}$$

(2) *(Symmetry Formula)*

$$V_k^n = V_j^n, \text{ where } k + j \equiv n \pmod 5.$$

(3) *(Initial Conditions)*

(i) For $n < 5$, $V_k^n = \binom{n}{k}$,

(ii) $V_0^5 = 2$, $V_k^5 = \binom{5}{k}$ for $k \in \{1, 2, 3, 4\}$.

Proof: To prove (1) let W^n be a set of size n and let W_k^n denote the collection of all subsets of W^n of size congruent to k (mod 5), $k \in \{0, 1, 2, 3, 4\}$. Then clearly $\#W_k^n = V_k^n$. Now let $w \in W^n$, $W \in W_k^n$. If $w \in W$, the remaining elements of W can be chosen from the $n - 1$ elements of $W^n - \{w\}$ in V_{k-1}^{n-1} ways. Otherwise, if $w \notin W$, the elements of W can be chosen from the $n - 1$ elements of $W^n - \{w\}$ in V_k^{n-1} ways.

To prove (2) let $n \in Z^+$, $k \in \{0, 1, 2, 3, 4\}$. The division algorithm yields integers m and j such that

$$n - k = 5m + j \quad \text{where } j \in \{0, 1, 2, 3, 4\},$$

and hence $k + j \equiv n$ (mod 5). Using this we can relate V_k^n and V_j^n as follows:

$$\begin{aligned} V_k^n &= \binom{n}{k} + \binom{n}{k+5} + \cdots + \binom{n}{k+5m} \\ &= \binom{n}{k} + \binom{n}{k+5} + \cdots + \binom{n}{n-j} \\ &= \binom{n}{n-k} + \quad \cdots \quad + \binom{n}{j} \\ &= \binom{n}{j+5m} + \quad \cdots \quad + \binom{n}{j} = V_j^n. \end{aligned}$$

The proof of (3) is trivial and so omitted. □

Using the initial conditions and the recursion for the V_k^n, we can build a table of values for the V_k^n similar to Pascal's triangle. Since the V_k^n are computed mod 5, there will be 5 entries in each row of our *Pascalian Rectangle*. In Figure 4 we illustrate the general form of the table and in Figure 5 we fill in specific values.

										Row
V_3^1		V_4^1		V_0^1		V_1^1		V_2^1		1
	V_4^2		V_0^2		V_1^2		V_2^2		V_3^2	2
V_4^3		V_0^3		V_1^3		V_2^3		V_3^3		3
	V_0^4		V_1^4		V_2^4		V_3^4		V_4^4	4
V_0^5		V_1^5		V_2^5		V_3^5		V_4^5		5
				⋮						⋮

FIGURE 4

Remark: Notice the wrap-around property of the table. The right-most entry in an even row (or the left-most entry in an odd row) is the sum of the left-most and right-most entries of the previous row, e.g.,

$$V_0^5 = V_4^4 + V_0^4 \quad \text{and} \quad V_4^4 = V_4^3 + V_3^3.$$

If the table is constructed as in Figure 4 above and Figure 5 below, then the left-most entry of the n^{th} row is next seen to be a smallest entry of the n^{th} row. Recalling the definition of $V(n)$, we state the following theorem.

										Row
0		0		1		1		0		1
	0		1		2		1		0	2
0		1		3		3		1		3
	1		4		6		4		1	4
2		5		10		10		5		5
	7		15		20		15		7	6
14		22		35		35		22		7
	36		57		70		57		36	8
72		93		127		127		93		9
					⋮					⋮

FIGURE 5

Theorem 1: For $n \in Z^+$,

$$V(n) = V^n_{k(n)}, \quad \text{with } k(n) = \left\lfloor \frac{n}{2} \right\rfloor - 2 \pmod 5.$$

Proof: That the left-most entry of the n^{th} row is of the form $[n/2] - 2$ follows from the recursion formula and induction. Next, we must show that the left-most entry of each row in Figures 4 and 5 is also a smallest entry of that row. This follows easily by induction once we verify that the symmetry of each row is maintained. But this is immediate from the symmetry formula of the Lemma. If n is even, then

$$\left\lfloor \frac{n}{2} \right\rfloor - 2 = \frac{n}{2} - 2.$$

If the left-most entry of the n^{th} row is $V^n_{(n/2)-2}$, then the right-most entry is of the form

$$V^n_{(n/2)-2+4} = V^n_{(n/2)+2}.$$

Since

$$\left(\frac{n}{2} - 2\right) + \left(\frac{n}{2} + 2\right) \equiv n \pmod 5$$

we have, by the Lemma, that

$$V^n_{(n/2)-2} = V^n_{(n/2)+2}.$$

Similarly,

$$\left(\frac{n}{2} - 2 + 1\right) + \left(\frac{n}{2} - 2 + 3\right) \equiv n \pmod 5$$

so that the second and fourth entries of the row are equal. Similar reasoning verifies the shifted row symmetry for n odd. An easy induction completes the proof. □

4. A Recursion for $V^n_{k(n)}$

Our next theorem provides a recursion and closed formula for $V(n)$.

Theorem 2: For any integer $n \in Z^+$,

$$\text{(i)} \quad V(n) = \begin{cases} 2V(n-1), & n \text{ odd}, \\ 2V(n-1) + F_{n-2}, & n \text{ even}. \end{cases}$$

(ii) $$V(n) = \begin{cases} \frac{1}{5} \cdot 2^n - \frac{2}{5} L_n, & n \text{ odd}, \\ \frac{1}{5} \cdot 2^n - \frac{1}{5} L_{n+1}, & n \text{ even}. \end{cases}$$

Proof: By the established symmetry of the table in Figure 5, the first and last entries in an even row are identical. Also, for n odd, we have

$$k(n) = \left\lfloor \frac{n}{2} \right\rfloor - 2 = \left\lfloor \frac{n-1}{2} \right\rfloor - 2 = k(n-1).$$

Therefore, we have, for n odd

(3) $$V^n_{k(n)} = V^{n-1}_{k(n)} + V^{n-1}_{k(n)-1} = V^{n-1}_{k(n-1)} + V^{n-1}_{k(n-1)-1} = 2V^{n-1}_{k(n-1)}.$$

For n even, in $V^n_{k(n)}$, we need to take a somewhat less direct approach. To this end, we define $D(n)$, for all n, as follows

(4) $$D(n) = \begin{cases} V^n_{k(n)+2} - V^n_{k(n)+1} & n \text{ odd}. \\ V^n_{k(n)+1} - V^n_{k(n)} & n \text{ even}. \end{cases}$$

We will show that $D(n) = F_n$. To begin with, consultation of Figure 5 verifies that

$$D(1) = 1 - 0 = 1,\ D(2) = 1 - 0 = 2,\ D(3) = 3 - 1 = 2,\ D(4) = 4 - 1 = 3.$$

Now, for n even, we have

$$\begin{aligned} D(n) &= V^n_{k(n)+1} - V^n_{k(n)} \\ &= [V^{n-1}_{k(n)+1} + V^{n-1}_{k(n)}] - [V^{n-1}_{k(n)} + V^{n-1}_{k(n)-1}] \\ &= [V^{n-1}_{k(n-1)+2} + V^{n-1}_{k(n-1)+1}] - [V^{n-1}_{k(n-1)+1} + V^{n-1}_{k(n-1)}] \\ &= [V^{n-1}_{k(n-1)+2} - V^{n-1}_{k(n-1)+1}] + [V^{n-1}_{k(n-1)+1} - V^{n-1}_{k(n-1)}] \\ &= D(n-1) + [V^{n-2}_{k(n-2)+1} - V^{n-2}_{k(n-2)}] \\ &= D(n-1) + D(n-2). \end{aligned}$$

A similar argument shows that the recursion holds for n odd. Since $D(n)$ satisfies the same recursion as F_n and the initial conditions are the same, we have that $D(n) = F_n$.

We return now to $V^n_{k(n)}$. For n even, we have

(5) $$\begin{aligned} V^n_{k(n)} &= V^{n-1}_{k(n)} + V^{n-1}_{k(n)-1} \\ &= V^{n-1}_{k(n-1)+1} + V^{n-1}_{k(n-1)} \\ &= 2V^{n-1}_{k(n-1)} + [V^{n-1}_{k(n-1)+1} - V^{n-1}_{k(n-1)}] \\ &= 2V^{n-1}_{k(n-1)} + [V^{n-2}_{k(n-2)+1} - V^{n-2}_{k(n-2)}] \\ &= 2V^{n-1}_{k(n-1)} + D(n-2) = 2V^{n-1}_{k(n-1)} + F_{n-2}. \end{aligned}$$

Combining the results in (3) and (5) yields

(6) $$V^n_{k(n)} = \begin{cases} 2V^{n-1}_{k(n-1)}, & n \text{ odd}, \\ 2V^{n-1}_{k(n-1)} + F_{n-2}, & n \text{ even}. \end{cases}$$

To solve this recursion we note that $[x/(1 - x - x^2)]$ is the generating function for the sequence F_0, F_1, F_2, ..., so that $[-x/(1 + x - x^2)]$ is the generating function for the sequence F_0, $-F_1$, F_2, $-F_3$, ..., and therefore

$$\frac{1}{2}\left[\frac{x}{1 - x - x^2} - \frac{x}{1 + x - x^2}\right]$$

is the generating function for the sequence F_0, 0, F_2, 0, F_4, ... Let

$$V(x) = \sum_{n \geq 1} V^n_{k(n)} x^n.$$

then (6) gives

$$V(x) = 2xV(x) + x - x^2 - x^3 - 2x^4 + \frac{1}{2}\left[\frac{x^2}{1 - x - x^2} - \frac{x^2}{1 + x - x^2}\right].$$

A partial fraction expansion of the rational function $V(x)$ leads, after some calculation, to the closed form:

$$(7) \qquad V^n_{k(n)} = \begin{cases} \frac{1}{5}\cdot 2^n - \frac{1}{5}L_{n+1}, & n \text{ even}, \\ \frac{1}{5}\cdot 2^n - \frac{2}{5}L_n, & n \text{ odd}. \end{cases}$$

Combining the results of (6) and (7) with the definition of $V(n)$ completes the proof. □

Corollary: Let $f(n, k)$ denote the largest order of an induced subgraph of Q_n that contains no sub-Q_4. Then

$$f(n, 4) \geq \begin{cases} \frac{4}{5}\cdot 2^n + \frac{1}{5}L_{n+1}, & n \text{ even}. \\ \frac{4}{5}\cdot 2^n + \frac{2}{5}L_n, & n \text{ odd}. \end{cases}$$

Proof: This follows from Theorem 2 and the fact that $f(n, 4) \geq 2^n - V(n)$. □

Remarks: (1) Recalling that $V^n_{k(n)}$ is a sum of binomial coefficients, it is interesting to observe the locations of these binomial coefficients in Pascal's triangle. In Figure 6, the circled entries in the n^{th} row of Pascal's triangle are the binomial coefficients that sum to $V^n_{k(n)}$. Observe that the circled entries are "as far as possible" from the binomial coefficients of the form $\binom{n}{n/2}$ [6].

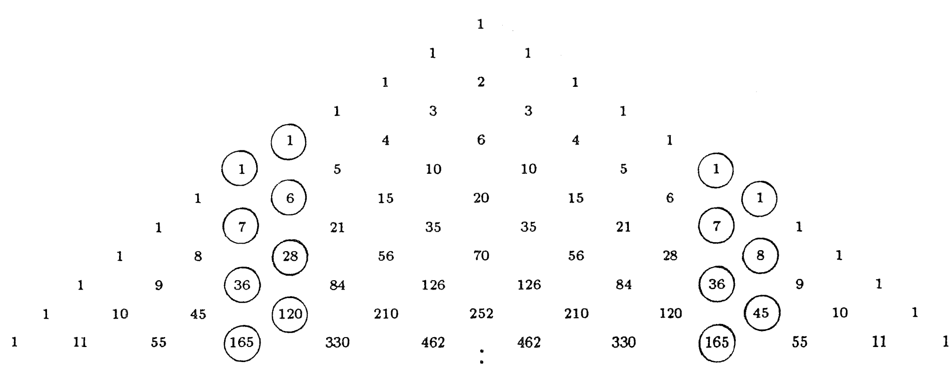


FIGURE 6

(2) A related problem appeared in the 35th W. L. Putnam Intercollegiate Mathematical Competition [7]; that problem asked for a calculation of S_k^n, where

$$S_k^n = \sum_{\substack{j \equiv k \\ \text{mod } 3}} \binom{n}{j}, \; k = 0, 1, 2.$$

5. Conclusion

By defining the terms V_k^n and $V(n)$ modulo 5, we were able to obtain an improved lower bound for $f(n, 4)$, the largest order of an induced subgraph of Q_n that contains no sub-Q_4. In general, by working modulo m, we can improve on the inequality (1) for $k = m - 1$; for $k \in \{0, 1, \ldots, m - 1\}$, let

$$V_{k,m}^n = \sum_{\substack{j \equiv k \\ \text{mod } m}} \binom{n}{j} \quad \text{and} \quad V(n, m) = \min_{0 \le k \le m-1} V_{k,m}^n.$$

Then $f(n, m - 1) \ge 2^n - V(n, m)$. Work on determination of $V(n, m)$, for all $m \le \{0, 1, \ldots, n\}$ is in progress by this author. It was originally conjectured that $f(n, m - 1) = 2^n - V(n, m)$ but this is now known to be true only for $m \in \{0, 1, 2\}$ [8].

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