EQUAL SUMS OF UNLIKE POWERS

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1. Introduction

Solutions are given for the Diophantine equation

$$x_1^p + x_2^p + \cdots + x_m^p = y_1^q + y_2^q + \cdots + y_n^q,\ p > 0,\ q > 0,\ m + n > 2,$$

for which we use the notation $(p.q.m.n)$. In a previous paper [1] we surveyed solutions of this equation for $p = q$ with p and $q \le 10$. We now show that $(p.q.m.n)$ has nontrivial parametric solutions in which the number of terms m, n on both sides of the equation depend on p and q. Some of these solutions will be valid when $p = q$ as a special case, but in general we assume that $p > q$. That is, we always write the equation with the higher exponent on the left-hand side. We assume that none of the x_i or y_j is zero, and that $x_i^p \ne y_j^q$, i.e., that equal individual terms on both sides of the equation have been removed. Rarely does this condition invalidate one of the many solutions available by our algorithms.

Related work includes a number of parametric solutions and also numerical solutions, usually involving low values of either p or q or both. Uspenski [2] gives a general solution in relatively prime integers of $z^n = x^2 + y^2$ for $n > 1$. Various solutions of the equation $z^2 = x^3 + y^3$ by Euler, Hoppe, Thue, and Schwering are given in Dickson [3]. The equation $(3.2.n.1)$ was solved for various values of n by a number of investigators [4], [5]. Cunningham gave a procedure for solving $(2n.4.2.3)$ in [6]. Several writers solved $(4.2.m.n)$ for various values of m and n [7]. Some numerical examples of biquadrates as the sum of several cubes or squares are given in [8]. A parametric solution of (5.2.3.1) was obtained by Bouniakowsky [9]. Cunningham solved (8.2.6.1) in [10] and both (4.2.3.3) and (8.4.3.3) in [11]. Rignaux solved (6.2.2.2) in [12]. Killgrove [13] discussed the equation $x^n + y^m = z^k$ and gave a proof for a theorem of Lebesgue [19] which states that if $x^{2t} + y^{2t} = z^2$ has a nontrivial solution, then t is odd and $u^t + v^t = w^t$ has a nontrivial solution. Beerensson [14] proved that $x^n + y^n = z^m$ has infinitely many integer solutions if m, n are relatively prime, but did not present explicit solutions. In [20], Kelemen proved two theorems on conditions for the solvability and form of solutions of the general equation

$$a_1x_1^{k1} + a_2x_2^{k2} + \cdots + a_nx_n^{kn} = 0,$$

and gave examples.

2. Solution for all Positive Values of p, q

Theorem 1: The Diophantine equation

$$x_1^p + x_2^p + \cdots + x_m^p = y_1^q + y_2^q + \cdots + y_n^q, \tag{1}$$

where $p > 0$, $q > 0$, $m > 0$, $n > 0$, and $m + n > 2$, has a nontrivial parametric integer solution, as follows. If d is the greatest common divisor of p and q, this solution exists for all m, n such that

$$m = \sum_{k=2}^{r}(u_k + v_kk^d),\quad n = \sum_{k=2}^{r}(v_k + u_kk^d),$$

where r is any integer > 1 and, for $k = 2, 3, \ldots, r$, the u_k and v_k are arbitrary nonnegative integers not all zero.

Proof: Since d is the greatest common divisor of p and q, there exist positive integers A, B, C, D such that

(2) $$Ap - Bq = Cq - Dp = d.$$

Let $a_1, a_2, \ldots, a_s$ and $b_1, b_2, \ldots, b_t$ be arbitrary nonzero integers where $s > 1$ and $t > 1$, and let

(3) $$u = \sum_{k=1}^{s} a_k^p, \quad v = \sum_{k=1}^{t} b_k^q.$$

Then u^d, when expanded by multiplication, is the sum of s^d terms, each of which is the product of d numbers of the form a_k^p. Therefore, each term of u^d is of the form y^p, where y is an integer. Thus, we have

(4) $$u^d = \sum_{j=1}^{s^d} y_j^p,$$

where the y_j are all integers. Similarly, we have

(5) $$v^d = \sum_{j=1}^{t^d} z_j^q,$$

where the z_j are all integers. Then, from (2) and (4),

$$u^{Cq} = u^{Dp}u^{Cq-Dp} = u^{Dp}u^d = \sum_{j=1}^{s^d} u^{Dp}y_j^p,$$

so that

(6) $$(u^C)^q = \sum_{j=1}^{s^d} (y_j u^D)^p$$

is a nontrivial parametric solution of (1) with $m = s^d$, $n = 1$, and having $s > 1$ arbitrary nonzero integer parameters $a_1, a_2, \ldots, a_s$. Similarly,

$$v^{Ap} = v^{Bq}v^{Ap-Bq} = v^{Bq}v^d = \sum_{j=1}^{t^d} v^{Bq}z_j^q,$$

or

(7) $$(v^A)^p = \sum_{j=1}^{t^d} (z_j v^B)^q,$$

which is a nontrivial parametric solution of (1) with $m = 1$, $n = t^d$, and having $t > 1$ arbitrary nonzero integer parameters $b_1, b_2, \ldots, b_t$.

Next, we may "add" two or more solutions of (1) by summing the terms with exponent p to form the left-hand side of the new solution and summing the terms with exponent q to form the right-hand side. Therefore, a valid nontrivial parametric solution of (1) may be obtained by summing u_k solutions of the form given by (7) for $t = k$, together with v_k solutions of the form given by (6) with $s = k$, where k takes on the values $2, 3, \ldots, r$ for any arbitrary integer $r > 1$. The numbers of solutions to be "added" in this way, u_k and v_k, may be any nonnegative integers not all zero. Then m, n, the number of terms in the resultant equation having exponents p, q, respectively, will be as given in the theorem.

Example 1: Let $p = 4$ and $q = 3$ so that $d = 1$. Take $A = B = 1$, $C = 3$, $D = 2$. Let $r = 2$ so that $s = 2$ and $t = 2$. We have

$$u = a_1^4 + a_2^4,\ v = b_1^3 + b_2^3,\ y_1 = a_1,\ y_2 = a_2,\ z_1 = b_1,\ z_2 = b_2.$$

The solution (6) becomes

$$(6.1)\qquad [(a_1^4 + a_2^4)^3]^3 = [a_1(a_1^4 + a_2^4)^2]^4 + [a_2(a_1^4 + a_2^4)^2]^4$$

and the solution (7) becomes

$$(7.1)\qquad (b_1^3 + b_2^3)^4 = [b_1(b_1^3 + b_2^3)]^3 + [b_2(b_1^3 + b_2^3)]^3.$$

Two numerical examples of (6.1) for $(a_1, a_2) = (1, 1)$ and $(2, 1)$ are

$$8^3 = 4^4 + 4^4;\ 4913^3 = 578^4 + 289^4.$$

Two numerical examples of (7.1) for $(b_1, b_2) = (2, 1)$ and $(3, 2)$ are

$$9^4 = 18^3 + 9^3;\ 35^4 = 105^3 + 70^3.$$

We may obtain further solutions by combining (through "addition") any number of the individual solutions. For example, from those given, we get

$$9^4 + 4^4 + 4^4 = 18^3 + 9^3 + 8^3;\ 578^4 + 289^4 + 35^4 = 4913^3 + 105^3 + 70^3,$$

and so on.

Example 2: Let $p = 6$ and $q = 4$ so that $d = 2$. Take $A = B = 1$. Set $r = 2$ so that $t = 2$. Then we have $v^2 = (b_1^4 + b_2^4)^2$, so that

$$z_1 = b_1^2,\ z_2 = z_3 = b_1b_2,\ z_4 = b_2^2.$$

Solution (7) becomes)

$$(7.2)\qquad (b_1^4 + b_2^4)^6 = [b_1^2(b_1^4 + b_2^4)]^4 + 2[b_1b_2(b_1^4 + b_2^4)]^4 + [b_2^2(b_1^4 + b_2^4)]^4.$$

Two numerical examples of (7.2) for $(b_1, b_2) = (1, 1)$ and $(2, 1)$ are

$$2^6 = 2^4 + 2^4 + 2^4 + 2^4;\ 17^6 = 68^4 + 34^4 + 34^4 + 17^4.$$

Note that the terms in each equation of the type (6) and (7) are not relatively prime. However, since the exponents p and q are different, it is not usually possible to remove a common factor and still have an equation remaining with the same exponents p and q. This would be possible if in equation (1) there is a divisor F of all the terms $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$, where F is of the form z^f and f is divisible by p and q, and $z > 1$. When solutions involving different sets of parameters a_i and b_j are combined by "addition," the resultant solution will not in general have such a common divisor (as in the examples given above).

3. Solution for p and q Relatively Prime

Theorem 2: Whenever p and q are relatively prime, equation (1) of Theorem 1 has a nontrivial parametric integer solution for all positive values of m, n such that $m + n > 2$.

Proof: In Theorem 1, let $d = 1$. We use the notation $(p.q.m.n)$ to denote equation (1). Then (6) gives a solution of $(p.q.s.1)$ for arbitrary $s > 1$, which we denote by (S). If $n = 1$, set $s = m$ to solve $(p.q.m.n)$ with m integer parameters. Similarly, (7) gives a solution of $(p.q.1.t)$ for arbitrary $t > 1$, which we denote by (T). If $m = 1$, set $t = n$ to solve $(p.q.m.n)$ with n integer parameters. Next, assume that $m > 2$ and $n > 2$. Now set $s = m - 1$ and $t = n - 1$. Then "add" the two solutions (S), (T) to obtain a new solution of $(p.q.s + 1.t + 1) = (p.q.m.n)$. This solution will have $s + t = m + n - 2$ arbitrary integer parameters. Next, if $m = 2$ and $n > 3$, add solution (T) with $t = 2$ to solution (T) with $t = n - 2$ to obtain a solution of $(p.q.2.n)$ having n integer parameters. Similarly, if $n = 2$ and $m > 3$, add solution (S) with $s = 2$ to solution (S) with $s = m - 2$ to obtain a solution of $(p.q.m.2)$ having m integer parameters.

There remain only three cases, namely, $(p.q.2.2)$, $(p.q.2.3)$, and $(p.q.3.2)$. For the case $m = n = 2$, let a, b be distinct positive integers, arbitrary except that both are even or both are odd. Then $a^q + b^q = 2w$, where w is an integer. Then, since p and q are relatively prime, we have $Ap - Bq = 1$ for integers A, B and

$$w^{Bq}(a^q + b^q) = w^{Bq}(2w) = 2w^{Bq+Ap-Bq} = 2w^{Ap}.$$

Then

$$(aw^B)^q + (bw^B)^q = (w^A)^p + (w^A)^p$$

is a solution of $(p.q.2.2)$ having two integer parameters a, b of equal parity but otherwise arbitrary. For the case $m = 2$, $n = 3$, let a, b, and c be distinct positive integers, arbitrary except that the sum $a^q + b^q + c^q = 2w$, where w is an integer. This can be achieved by selecting a, b, and c to all be even, or by choosing one of a, b, or c to be even and the others odd.

Then, as before, we have $Ap - Bq = 1$ for integers A, B, and

$$w^{Bq}(a^q + b^q + c^q) = w^{Bq}(2w) = 2w^{Bq+Ap-Bq} = 2w^{Ap}.$$

Therefore,

$$(aw^B)^q + (bw^B)^q + (cw^B)^q = (w^A)^p + (w^A)^p$$

is a solution of $(p.q.2.3)$ having three integer parameters. In a similar manner, we can generate a three-parameter solution of $(p.q.3.2)$. This completes the proof.

Example 3: Let $p = 8$ and $q = 5$. First, to solve $(8.5.2.2)$, take $a = 3$, $b = 1$ so that $3^5 + 1^5 = 244 = 2(122)$ and $w = 122$. Then, since $2(8) - 3(5) = 1$, we may take $A = 2$, $B = 3$, and $122^{15}(3^5 + 1^5) = 122^{16}(2)$, or

$$[3(122)^3]^5 + [(122^3)]^5 = [(122^2)]^8 + [(122^2)]^8.$$

To solve $(8.5.2.3)$, take $a = 2$, $b = c = 1$, so that $2^5 + 1^5 + 1^5 = 34 = 2(17)$ and $w = 17$. Then, $17^{15}(2^5 + 1^5 + 1^5) = 17^{16}(2)$, or

$$[2(17^3)]^5 + (17^3)^5 + (17^3)^5 = (17^2)^8 + (17^2)^8.$$

4. Derived Solutions

Theorem 3: If a specific nontrivial solution of equation $(p.q.m.n)$ exists for which all of the n terms y_j^q in equation (1) are equal, then a nontrivial solution exists for the equation $(q + pr.p.n.m)$, where r is any nonnegative integer.

Proof: If

$$nb^q = \sum_{i=1}^{m} a_i^p$$

is the specific nontrivial solution of $(p.q.m.n)$, then

$$nb^q b^{pr} = b^{pr}\sum_{i=1}^{m} a_i^p = \sum_{i=1}^{m} (a_i b^r)^p = nb^{q+pr}$$

is a solution of the equation $(q + pr.p.n.m)$.

Example 4: A computer search by the author yielded the smallest nontrivial solution of (6.2.3.1) as $100^6 + 81^6 + 42^6 = 1134865^2$. If we set $b = 1134865$, we have

$$(100b^r)^6 + (81b^r)^6 + (42b^r)^6 = b^{6r+2}$$

as a solution of equation $(6r + 2.6.1.3)$ for $r \geq 0$.

Theorem 3 can also be applied when $p = q$. The solutions recently found by Eklies [15] and Frye [16] to the equation $x^4 + y^4 + z^4 = t^4$ allows us to solve the equation $(4r + 4.4.1.3)$, for any integer $r \geq 0$. In particular, for $r = 1$, we have

$$(tx)^4 + (ty)^4 + (tz)^4 = t^8$$

as a solution of (8.4.1.3), where $x = 95800$, $y = 217519$, $z = 414560$, and $t = 422481$. Other solutions to the equation $(p.p.m.n)$ can be found in [1].

5. Incompleteness of the Theorems

The solutions to (1) produced by the algorithms of Theorems 1, 2, and 3 are not complete. The smallest nontrivial solution of (4.2.3.1) is

$$20^4 + 15^4 + 12^4 = 481^2,$$

which cannot be produced by Theorem 1, since 481 is prime to 20, 15, and 12. The smallest nontrivial solution of (4.3.2.2) is

$$11^4 + 8^4 = 24^3 + 17^3.$$

This solution cannot be produced by Theorem 2, which yields only solutions of the form

$$x_1^p + x_2^p = 2y_1^q,$$

or by Theorem 3, which yields only solutions of the form

$$x_1^p + x_2^p + \cdots + x_m^p = ny_1^q.$$

6. Table of Solutions

We supplement the discussion by presenting in Table 1 a list of solutions to equation $(p.q.m.n)$ for p and $q < 10$ and m and $n < 4$. The solutions were obtained by a combination of methods, including the use of Theorems 1, 2, and 3, computer search, and reference to the literature. As illustrated in the table, the solutions produced by use of Theorems 1, 2, and 3 are incomplete, since solutions exist for which the terms in (1) have no common divisor > 1. Table 1 lists the solutions in smallest integers known to the author. Some equations have no nontrivial solutions. The equations (6.3.1.2), (6.3.2.1), (9.3.1.2), (9.3.2.1), (9.6.1.2), and (9.6.2.1) have no nontrivial solution because, as Euler proved [17], the equation $x^3 + y^3 = z^3$ has no solution with $xy \neq 0$; similarly, equations (4.2.2.1), (6.4.1.2), (8.2.2.1) and (8.6.2.1) cannot be solved nontrivially because Euler showed that the equation $x^4 + y^4 = z^2$ has no solution with $xy \neq 0$ [18]. The equations (6.2.2.1), (6.4.2.1), and (8.6.1.2) are impossible (because $x^3 + y^3 = z^3$ is impossible) by a theorem of Lebesgue [19]. As shown in Table 1, the equations for small values of p, q, m, and n which appear to be the most difficult to solve in small integers are (6.3.3.2), (6.3.3.1), (6.4.2.2), (6.4.3.1), (6.4.3.2), (8.2.3.1), (8.4.2.2), (8.4.3.1), (8.4.3.2); $(8.6.m.n)$ for $m < 4$, $n < 4$ except (8.6.1.3); (9.3.3.1); and $(9.6.m.n)$ with $m < 4$, $n < 4$. Although solutions were not found for these specific values of p, q, m, n, we can obtain solutions for the same values of p, q with larger values of m, n by applying Theorem 1. For example, solutions for (9.6.1.8) and (9.6.8.1) are

$$u^9 = (a^3u)^6 + (b^3u)^6 + 3[(a^2bu)^6 + (ab^2u)^6], \quad u = a^6 + b^6;$$

$$(v^2)^6 = (a^3v)^9 + (b^3v)^9 + 3[(a^2bv)^9 + (ab^2v)^9], \quad v = a^9 + b^9,$$

where a and b are arbitrary integers. If $a = 2$ and $b = 1$, then $u = 65$ and $v = 513$ and these solutions become

$$65^9 = 520^6 + 3(260^6) + 3(130^6) + 65^6;$$

$$263169^6 = 4104^9 + 3(2052^9) + 3(1026^9) + 513^9.$$

The author would be pleased to receive correspondence concerning any new solutions to the equations discussed above.

TABLE 1. Solutions of $\sum_{i=1}^{m} x_i^p = \sum_{j=1}^{n} y_j^q$

Legend: The entry $x_1, x_2, \ldots, x_m = y_1, y_2, \ldots, y_n$ denotes the solution.

p.q \ m.n	1.2	1.3	2.1	2.2
3.2	2=2,2 5=10,5 5=11,2	3=3,3,3 3=5,1,1	2,1=3 2,2=4 8,4=24	4,1=7,4 4,2=6,6
4.2	5=24,7 5=20,15	3=6,6,3 3=7,4,4	Impossible	5,5=35,5
4.3	2=2,2 9=18,9	3=3,3,3	4,4=8 32,32=128 108,108=648	11,8=24,17 14,14=42,14
5.2	2=4,4 5=41,38	3=9,9,9 3=11,11,1	2,2=8 8,8=256	3,1=12,10 4,1=31,8
5.3	3=6,3 4=8,8	6=18,12,6 9=27,27,27	2,2=4	6,6=24,12 12,10=70,18
5.4	2=2,2	3=3,3,3	8,8=16	41,41=123,41
6.2	5=100,75 5=117,44 5=120,35	3=18,18,9 3=26,7,2	Impossible	2,1=7,4 3,1=21,17
6.3	Impossible	3=8,6,1 5=22,17,4 6=30,24,18	Impossible	18,12=330,102 172,86=27778,16942
6.4	Impossible	481=20(481), 15(481),12(481)	Impossible	Unknown
6.5	2=2,2 33=66,33	3=3,3,3 34=68,34,34	16,16=32	122,122=366,122
7.2	2=8,8 5=205,190 5=250,125 5=278,29	3=45,9,9 3=43,17,17	2,2=16 8,8=2048	4,1=127,16 4,1=103,76 4,1=92,89
7.3	2=4,4 9=162,81	3=9,9,9 6=64,26,6	4,4=32 32,32=4096	14,14=588,196 16,12=620,404
7.4	8=32,32	11=55,55,33 27=243,243,243	2,2=4	$41^3,41^3=3(41^5),41^5$
7.5	8=16,16	27=81,81,81	4,4=8	$122^3,122^3=$ $3(122^4),122^4$
7.6	2=2,2 65=130,65	3=3,3,3 66=132,66,66	32,32=64 $2a^5,a^5=a^6$ $a=129$	365,365=1095,365

TABLE 1 (continued)

p.q \ m.n	1.2	1.3	2.1	2.2
8.2	5=500,375 5=585,220 5=600,165	3=54,54,27 3=63,36,36 3=79,16,8	Impossible	3,1=71,39 3,2=79,24 4,3=264,49
8.3	3=18,9 4=32,32	6=108,72,36 8=255,57,22 9=243,243,243	2,2=8	6,6=144,72
8.4	Impossible	See Text, Section 4.	Impossible	Unknown
8.5	8,8=32 $33^2=2(33^2)$, 33^2	9=27,27,27	8,8=32	$a^2,a^2=3a^3,a^3$ a=122
8.6	Impossible	a=100a,81a, 42a a=1134865	Impossible	Unknown
8.7	2=2,2	3=3,3,3	64,64=128	a,a=3a,a a=1094
9.2	2=16,16	3=81,81,81	2,2=32	5,5=1875,625
9.3	Impossible	3=24,18,3 5=110,85,20	Impossible	4,2=60,36 7,2=322,191 8,4=480,288
9.4	2=4,4 17=578,289	3=9,9,9	8,8=128	$a,a=3a^2,a^2$ a=41
9.5	16=128,128	$81=3^7,3^7,3^7$	2,2=4	$a^4,a^4=3a^7,a^7$ a=122
9.6	Impossible	Unknown	Impossible	Unknown
9.7	16=32,32	$81=3^5,3^5,3^5$	8,8=16 $2a^3,a^3=a^4$ a=513	$a^4,a^4=3a^5,a^5$ a=1094
9.8	2=2,2 257=514,257	3=3,3,3	128,128=256	a,a=3a,a a=3281

TABLE 1 (continued)

p.q \ m.n	2.3	3.1	3.2	3.3
3.2	3,2=5,3,1 3,3=5,5,2	3,2,1=6 3,3,3=9 6,2,1=15	3,1,1=5,2 4,2,1=8,3	2,2,1=3,2,2
4.2	2,1=3,2,2 3,1=8,3,3	20,15,12=481	2,1,1=3,3 3,2,1=7,7	2,2,1=5,2,2 3,1,1=7,5,3
4.3	5,4=9,5,3	5,5,3=11 9,9,9=27	3,3,3=6,3 8,5,4=17,4	4,1,1=5,5,2
5.2	2,1=5,2,2 3,1=12,8,6 3,3=22,1,1	3,3,3=27 12,12,12=864 15,5,5=875	2,1,1=5,3 2,2,1=7,4	2,1,1=4,3,3 2,2,1=6,5,2
5.3	6,4=17,15,8 8,3=32,6,3 9,3=36,21,15	3,3,3=9 24,24,24=288 68,34,34=1156	9,3,3=39,6 9,9,9=54,27	3,2,2=6,4,3 3,3,3=8,6,1
5.4	9,9=18,9,9	27,27,27=81	17,4,1=37,17	6,4,3=9,7,3 6,6,6=12,6,6
6.2	2,1=6,5,2 5,3=127,12,9 5,5=176,15,7	100,81,42= 1134865	3,2,1=25,13 3,2,2=29,4 3,3,2=39,1	2,1,1=5,5,4 2,2,1=10,5,2 2,2,1=11,2,2
6.3	7,5=46,33,1 7,6=50,34,1	Unknown	Unknown	3,3,1=11,4,4 6,2,1=30,25,16
6.4	3,3=6,3,3 7,7=19,18,1 7,7=21,14,7	Unknown	Unknown	10,6,1=30,22,7 10,9,1=34,21,5
6.5	17,17=34,17, 17	81,81, 81,=243	11,11,11=22,11	16,16,2=32,2,2
7.2	2,1=11,2,2 2,1=10,5,2 2,1=8,7,4	3,3,3=81 12,12,12= 10368	2,1,1=11,3 2,1,1=9,7	2,2,1=15,4,4 2,2,1=12,8,7 2,2,2=16,8,8
7.3	4,2=20,20,8 5,4=44,21,4	9,9,9=243	3,3,1=15,10 3,3,3=18,9	4,4,2=32,4,4 6,6,2=76,49,15

TABLE 1 (continued)

p.q \ m.n	2.3	3.1	3.2	3.3
7.4	9,9=54,27,27	3,3,3=9	$a^3,a^3,a^3=6a^5,3a^5$ a=459	6,1,1=23,3,2 8,2,2=32,32,4
7.5	$a^3,a^3=2a^4$, a^4,a^4 a=17	9,9,9=27	$2a^2,a^2,a^2=a^3,a^3$ a=65	8,4,4=16,16,8
7.6	33,33=66, 33,33	243,243,243 =729	a,a,a=6a,3a a=15795	32,32,2=64,2,2
8.2	2,1=11,10,6 2,1=12,8,7	Unknown	3,3,2=97,63	2,1,1=11,11,4 2,1,1=13,18,5
8.3	4,2=40,12,4 4,2=33,31,4	3,3,3=27	9,9,9=486,243	2,1,1=5,5,2 3,2,2=18,9,8
8.4	7,7=56,35,21 7,7=55,39,16	Unknown	Unknown	4,4,3=18,13,8 5,4,3=24,19,5
8.5	$a^2,a^2=2a^3$, a^3,a^3 a=17	27,27,27=243	$11^2,11^2,11^2=2(11^3),11^3$	$a^2,a^2,a^2=3a^3$, $2a^3,a^3$ a=92
8.6	Unknown	Unknown	Unknown	Unknown
8.7	65,65=130, 65,65	$3^6,3^6,3^6=3^7$	43,43,43=86,43	a,a,a=3a,2a,a a=772
9.2	3,3=162, 81,81	3,3,3=243	2,1,1=17,15 $2a,a,a=a^5,a^5$ a=257	2,2,2=32,16,16
9.3	4,2=57,42,15 4,3=65,19,7	Unknown	12,8,8=1808,-784	3,2,1=23,18,3 3,3,2=32,17,13
9.4	9,9=162,81, 81	$27,27,27=3^7$	$a,a,a=6a^2,3a^2$ a=459	8,8,2=128,4,4
9.5	$a^4,a^4=2a^7$, a^7,a^7 a=17	3,3,3=9	$2a,a,a=a^2,a^2$ a=257	$3a,2a,a=a^2,a^2,a^2$ a=6732
9.6	Unknown	Unknown	Unknown	Unknown
9.7	$a^4,a^4=2a^5$, a^5,a^5 a=65	27,27,27=81	$2a^3,a^3,a^3=a^4,a^4$ a=257	$2a^3,a^3,16=a^4,32$, 32 a=513
9.8	a,a=2a,a,a a=129	$3^7,3^7,3^7=3^8$	$2a^7,a^7,a^7=a^8,a^8$ a=257	128,128,2=256, 2,2

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