REPEATING DECIMALS REPRESENTED BY TRIBONACCI SEQUENCES APPEARING FROM LEFT TO RIGHT OR FROM RIGHT TO LEFT

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1. Introduction

In 1953 Fenton Stancliff [1] noted that

$$\sum 10^{-(i+1)} F_i = \frac{1}{89},$$

where F_i denotes the i^{th} Fibonacci number. This curious property of Fibonacci numbers attracts many Fibonacci fanciers. Afterward, Long [2], Hudson & Winans [3], Winans [4], and Lin [5] discussed this Fibonacci phenomenon from different viewpoints. Köhler [6] and Hudson [7] then discussed Tribonacci series decimal expansions. In Lin [8], the characteristics of four types of Tribonacci series

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \text{ where } T_1 = 1,\ T_2 = 1,\ T_3 = 2,$$

$$R_n = R_{n-1} + R_{n-2} + R_{n-3}, \text{ where } R_1 = 1,\ R_2 = 3,\ R_3 = 7,$$

$$S_n = S_{n-1} + S_{n-2} + S_{n-3}, \text{ where } S_1 = 2,\ S_2 = 5,\ S_3 = 10,$$

$$U_n = U_{n-1} + U_{n-2} + U_{n-3}, \text{ where } U_1 = 1,\ U_2 = 2,\ U_3 = 3,$$

are further explored in their $X^3 - X^2 - X - 1 = 0$ format. But, in Lin [8], there was a question left open, which is whether T_n, R_n, S_n, and U_n could be described as one of the four different types of decimal expansions represented by sequential Tribonacci series of the form:

A. $0.T_{n1}T_{n2}T_{n3}T_{n4}T_{n5}T_{n6}T_{n7}\ldots = N_a/M_a$,

B. $0.T_{n1}\overline{T}_{n2}T_{n3}\overline{T}_{n4}T_{n5}\overline{T}_{n6}T_{n7}\ldots = N_b/M_b$,

C. N_c/M_c ends in $\ldots T_{n7}T_{n6}T_{n5}T_{n4}T_{n3}T_{n2}T_{n1}$,

D. for $N_d/M_d > 0$, N_d/M_d ends in $\ldots T_{n7}\overline{T}_{n6}T_{n5}\overline{T}_{n4}T_{n3}\overline{T}_{n2}T_{n1}$,

for $N_d/M_d < 0$, N_d/M_d ends in $\ldots \overline{T}_{n7}T_{n6}\overline{T}_{n5}T_{n4}\overline{T}_{n3}T_{n2}\overline{T}_{n1}$,

where $\overline{T}_{nm} = -T_{nm}$.

The terms of decimal expansion A are all positive, and those of decimal expansion B appear positive and negative alternately. The repetends of C and D are viewed in retrograde fashion, reading from the rightmost digit of the repeating cycle toward the left. The terms of repetend C are all positive, and those of repetend D appear positive and negative alternately. This question has been given a positive answer in this article. In the following, each of those four types of decimal expansions will be explored.

2. Decimal Fractions That Can Be Represented in Terms of Tribonacci Series Reading from Left to Right

Summing the geometric progressions using the same method described in Lin [5], Köhler [6], and Hudson [7], we can easily obtain the decimal fractions of the Tribonacci series T_{nm+p} as equation (1).

Theorem 1:

$$(1)\qquad \sum_{m=1}^{\infty} \frac{T_{nm+p}}{10^{km}} = \frac{T_{n+p}\cdot 10^{2k} + (T_{2n+p} - R_n\cdot T_{n+p})\cdot 10^k + T_p}{10^{3k} - R_n\cdot 10^{2k} + R_{-n}\cdot 10^k - 1}.$$

R_{mn+p}, S_{mn+p}, and U_{mn+p} have the same representation if we change T into R, S, and U, respectively.

When $p = 0$, they become

$$(2)\qquad \sum_{m=1}^{\infty} \frac{T_{nm}}{10^{km}} = \frac{T_n\cdot 10^{2k} + (T_{2n} - T_n\cdot R_n)\cdot 10^k + T_0}{10^{3k} - R_n\cdot 10^{2k} + R_{-n}\cdot 10^k - 1},$$

$$(3)\qquad \sum_{m=1}^{\infty} \frac{R_{nm}}{10^{km}} = \frac{R_n\cdot 10^{2k} + (R_{2n} - R_n^2)\cdot 10^k + R_0}{10^{3k} - R_n\cdot 10^{2k} + R_{-n}\cdot 10^k - 1},$$

$$(4)\qquad \sum_{m=1}^{\infty} \frac{S_{nm}}{10^{km}} = \frac{S_n\cdot 10^{2k} + (S_{2n} - S_n\cdot R_n)\cdot 10^k + S_0}{10^{3k} - R_n\cdot 10^{2k} + R_{-n}\cdot 10^k - 1},$$

$$(5)\qquad \sum_{m=1}^{\infty} \frac{U_{nm}}{10^{km}} = \frac{U_n\cdot 10^{2k} + (U_{2n} - U_n\cdot R_n)\cdot 10^k + U_0}{10^{3k} - R_n\cdot 10^{2k} + R_{-n}\cdot 10^k - 1},$$

where n and k must satisfy

$$(6)\qquad \frac{1}{3\cdot 10^k}\left[R_n + \frac{S_{n-1}}{3}(X + Y) + \frac{T_{n-2}}{3}(X^2 + Y^2)\right] < 1,$$

where $X = \sqrt[3]{19 + 3\sqrt{33}}$ and $Y = \sqrt[3]{19 - 3\sqrt{33}}$. Also,

$$(7)\qquad R_{-n} = R_{-n+3} - R_{-n+2} - R_{-n+1}.$$

Some particular values for the above series are summarized in Tables 1-4.

TABLE 1. Some values of $\sum_{m=1}^{\infty} \frac{R_{nm}}{10^{km}}$

$k \backslash n$	1	2	3	4	5	6	7
1	$\frac{123}{889}$	$\frac{323}{689}$	$\frac{603}{349}$				
2	$\frac{10203}{989899}$	$\frac{30203}{969899}$	$\frac{69003}{930499}$	$\frac{111003}{889499}$	$\frac{210203}{789899}$	$\frac{387803}{611099}$	$\frac{713003}{288499}$
3	$\frac{1002003}{998998999}$	$\frac{3002003}{996998999}$	$\frac{6990003}{993004999}$	$\frac{11010003}{988994999}$	$\frac{21002003}{978998999}$	$\frac{38978003}{961010999}$	$\frac{71030003}{928984999}$

TABLE 2. Some values of $\sum_{m=1}^{\infty} \frac{S_{nm}}{10^{km}}$

$k \backslash n$	1	2	3	4	5	6	7
1	$\frac{233}{889}$	$\frac{523}{689}$	$\frac{893}{349}$				
2	$\frac{20303}{989899}$	$\frac{50203}{969899}$	$\frac{98903}{930499}$	$\frac{171203}{889499}$	$\frac{320103}{789899}$	$\frac{587603}{611099}$	$\frac{1083503}{288499}$
3	$\frac{2003003}{998998999}$	$\frac{5002003}{996998999}$	$\frac{9989003}{993004999}$	$\frac{17012003}{988994999}$	$\frac{32001003}{978998999}$	$\frac{58976003}{961010999}$	$\frac{108035003}{928984999}$

TABLE 3. Some values of $\sum_{m=1}^{\infty} \frac{T_{nm}}{10^{km}}$

$k \backslash n$	1	2	3	4	5	6	7
1	$\frac{100}{889}$	$\frac{110}{689}$	$\frac{190}{349}$				
2	$\frac{10000}{989899}$	$\frac{10100}{969899}$	$\frac{19900}{930499}$	$\frac{40000}{889499}$	$\frac{70200}{789899}$	$\frac{129700}{611099}$	$\frac{240100}{288499}$
3	$\frac{1000000}{998998999}$	$\frac{1001000}{996998999}$	$\frac{1999000}{993004999}$	$\frac{4000000}{988994999}$	$\frac{7002000}{978998999}$	$\frac{12997000}{961010999}$	$\frac{24001000}{928984999}$

TABLE 4. Some values of $\sum_{m=1}^{\infty} \frac{U_{nm}}{10^{km}}$

$k \backslash n$	1	2	3	4	5	6	7
1	$\frac{110}{889}$	$\frac{200}{689}$	$\frac{290}{349}$				
2	$\frac{10100}{989899}$	$\frac{20000}{969899}$	$\frac{29900}{930499}$	$\frac{60200}{889499}$	$\frac{109900}{789899}$	$\frac{199800}{611099}$	$\frac{370500}{288499}$
3	$\frac{1001000}{998998999}$	$\frac{2000000}{996998999}$	$\frac{2999000}{993004999}$	$\frac{6002000}{988994999}$	$\frac{10999000}{978998999}$	$\frac{19998000}{961010999}$	$\frac{37005000}{928984999}$

Using (6) and $k = 1, 2, 3$, $n = 4, 8, 12$, respectively, we obtain:

$$[11 + 10 \cdot 4.51786\ldots/3 + 2 \cdot 12.41106\ldots/3]/30 = 1.14445\ldots > 1;$$

$$[131 + 108 \cdot 4.51786\ldots/3 + 24 \cdot 12.41106\ldots/3]/300 = 1.30977\ldots > 1;$$

$$[1499 + 1238 \cdot 4.51786\ldots/3 + 274 \cdot 12.41106\ldots/3]/3000 = 1.49897\ldots > 1.$$

These indicate that the ratios of geometric progressions are greater than 1; thus, the sums are divergent. This explains all the blanks in Tables 1-4.

3. Decimal Fractions That Can Be Represented in Terms of Alternating Positive and Negative Tribonacci Series Reading from Left to Right

Long [2] gave a proof for

$$\sum_{m=1}^{\infty} \frac{F_{m-1}}{(-10)^m} = 1/109;$$

Lin [5] proved

$$\sum_{m=1}^{\infty} \frac{F_{nm}}{(-10^k)^{m+1}} = \frac{F_n}{10^{2k} + 10^k \cdot L_n + (-1)^n}$$

and

$$\sum_{m=1}^{\infty} \frac{L_{nm}}{(-10^k)^{m+1}} = \frac{L_n}{10^{2k} + 10^k \cdot L_n + (-1)^n},$$

where L_m is the m^{th} Lucas number. These equations show that Fibonacci and Lucas numbers appear as the positive and negative terms of alternated Fibonacci and Lucas series, viz.,

$$N/M = 0.\ F_1\overline{F}_2F_3\overline{F}_4F_5\overline{F}_6\ldots,$$

where $\overline{F}_m = -F_m$, and the F_m appears successively in the repetend in blocks of k digits. In this case of Tribonacci sequences, if we substitute (-10^k) for 10^k in equation (2), it will appear as:

$$\text{(8)} \qquad \sum_{m=1}^{\infty} \frac{-T_{nm}}{(-10^k)^m} = \frac{T_n \cdot 10^{2k} + (T_nR_n - T_{2n}) \cdot 10^k + T_0}{10^{3k} + R_n \cdot 10^{2k} + R_{-n} \cdot 10^k + 1}$$

Changing T into R, S, and U, it will still be true.

TABLE 5. Some particular values for the T_{mn} series

$k \backslash n$	1	2	3	4	5	6	7
1	$\frac{100}{1091}$	$\frac{90}{1291}$	$\frac{210}{1751}$				
2	$\frac{1000}{1009901}$	$\frac{9900}{1029901}$	$\frac{20100}{1070501}$	$\frac{40000}{1109501}$	$\frac{69800}{1209901}$	$\frac{130300}{1391101}$	$\frac{239900}{1708501}$
3	$\frac{1000000}{1000999001}$	$\frac{999000}{1002999001}$	$\frac{2001000}{1007005001}$	$\frac{4000000}{1010995001}$	$\frac{6998000}{1020999001}$	$\frac{13003000}{1039011001}$	$\frac{23999000}{1070985001}$

4. Decimal Fractions That Can Be Represented in Terms of Tribonacci Series Reading from Right to Left

Winans [4] pointed out that 1/109, 9/71, and 1/10099 can be expressed as a reverse diagonalization of sums of Fibonacci numbers reading from the far right on the repeating cycle, where 1/109 ends in

```
    13853211
   21
  34
 55
...
...8623853211
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Johnson [9] gave a short solution to this kind of problem. Summing from the rightmost digit of the repeating cycle toward the left, she got the result:

$$\text{(9)} \qquad \frac{F_n}{10^{2k} + L_n \cdot 10^k - 1}, \ n \text{ is odd.}$$

Summing the geometric progressions by using the Binet form for Tribonacci T_n as Lin did in [8], and using the method indicated in Johnson [9], for $k > 0$, we can derive:

$$\text{(10)} \qquad \sum_{m=1}^{L} 10^{k(m-1)} T_{nm} = \frac{[T_{n(L-1)} \cdot 10^{k(L+1)} + (T_{n(L+1)} - R_n T_{nL}) \cdot 10^{kL} + }{10^{3k} - R_{-n} \cdot 10^{2k} + R_n \cdot 10^k - 1}$$

$$\underline{T_{nL} \cdot 10^{k(L-1)} - T_0 \cdot 10^{2k} - (T_{2n} - R_n T_n) \cdot 10^k - T_n]}$$

Let the denominator be acronymed as M, and $L(M)$ be the length of the period of M. We add

$$[-T_0 \cdot 10^{2k} - (T_{2n} - R_n T_n) \cdot 10^k - T_n] \cdot 10^{L(M)}$$
$$+ [T_0 \cdot 10^{2k} + (T_{2n} - R_n T_n) \cdot 10^k + T_n] \cdot 10^{L(M)}$$

to the numerator and divide both sides of (10) by $10^{k(L(M))}$; then it becomes

$$\sum_{m=1}^{L} 10^{k(m-1-L(M))} T_{nm}$$
$$= \frac{T_{n(L-1)} \cdot 10^{k(L+1-L(M))} + (T_{n(L+1)} - R_n T_{nL}) \cdot 10^{k(L-L(M))}}{M}$$
$$+ \frac{T_{nL} \cdot 10^{k(L-1-L(M))} - T_0 \cdot 10^{2k} - (T_{2n} - R_n T_n) \cdot 10^k - T_n}{M}$$
$$+ \frac{(T_0 \cdot 10^{2k} + (T_{2n} - R_n T_n) \cdot 10^k + T_n)(10^{L(M)} - 1)}{M \cdot 10^{L(M)}}$$

and, we get

Theorem 2: The decimal representation of

$$\text{(11)} \qquad \frac{N}{M} = \frac{T_0 \cdot 10^{2k} + (T_{2n} - R_n T_n) \cdot 10^k + T_n}{10^{3k} - R_{-n} \cdot 10^{2k} + R_n \cdot 10^k - 1}, \ N > 0,$$

ends in successive terms of T_{mn}, $m = 1, 2, 3, \ldots$, reading from the right end of the repeating cycle and appearing in groups of k digits.

If $N < 0$, then we have

Theorem 3: The decimal representation of

$$\text{(12)} \qquad \frac{M + N}{M} = \frac{M + T_0 \cdot 10^{2k} + (T_{2n} - R_n T_n) \cdot 10^k + T_n}{10^{3k} - R_{-n} \cdot 10^{2k} + R_n \cdot 10^k - 1}$$

ends in successive terms of T_{mn}, $m = 1, 2, 3, \ldots$, reading from the right end of the repeating cycle and appearing in groups of k digits, if 1 is added to the rightmost digit.

Proof: If N is negative, the N/M still has a positive term there. The numerator needs to be adjusted as below:

$$\frac{(T_0 \cdot 10^{2k} + (T_{2n} - R_nT_n) \cdot 10^k + T_n)(10^{L(M)} - 1)}{10^{L(M)} \cdot M}$$

$$= \frac{(T_0 \cdot 10^{2k} + (T_{2n} - R_nT_n) \cdot 10^k + T_n)(10^{L(M)} - 1) + (10^{L(M)} - 1)M - (10^{L(M)} - 1)M}{10^{L(M)} \cdot M}$$

$$= \frac{(M + T_0 \cdot 10^{2k} + (T_{2n} - R_nT_n) \cdot 10^k + T_n)(10^{L(M)} - 1)}{10^{L(M)} \cdot M} + \frac{1}{10^{L(M)}} - 1$$

The fractional part represents $(M + N)/M$ times one cycle of the repetend of $1/M$, when 1 is added to the rightmost digit.

Using the same method, we derive (11) and (12), and we can further generalize them to

Theorem 4:

$$\frac{N}{M} = \frac{T_p \cdot 10^{2k} + (T_{2n+p} - R_nT_{n+p}) \cdot 10^k + T_{n+p}}{10^{3k} - R_{-n} \cdot 10^{2k} + R_n \cdot 10^k - 1}, \quad N > 0, \tag{13}$$

$$\frac{M + N}{M} = \frac{M + T_p \cdot 10^{2k} + (T_{2n+p} - R_nT_{n+p}) \cdot 10^k + T_{n+p}}{10^{3k} - R_{-n} \cdot 10^{2k} + R_n \cdot 10^k - 1}, \quad N < 0, \tag{14}$$

ends in T_{mn+p}, reading from the right end of the repeating cycle and appearing in groups of k digits. If $N < 0$, 1 is added to the rightmost digit.

From the above method, we can easily obtain the decimal fractions that end in successive terms of R_{mn+p}, S_{mn+p}, and U_{mn+p} by changing T into R, S, and U, respectively.

Tables 6-9 show some values of T_{mn+p}, R_{mn+p}, S_{mn+p}, and U_{mn+p}, for $p = -3$, -2, -1, 0, 1, 2, 3, and $n = 1, 2, 3, 4, 5$.

TABLE 6. Fractions whose repetends end with successive terms of $T_{mn \pm p}$, occurring in repeating blocks of one digit

$p \backslash n$	1	2	3	4	5
−3	$\frac{1000}{1109}$	$\frac{1039}{1129}$ ▲	$\frac{489}{569}$ ▲	$\frac{1470}{1609}$ ▲	$\frac{1240}{1309}$ ▲
−2	$\frac{100}{1109}$	$\frac{110}{1129}$	$\frac{71}{569}$	$\frac{121}{1609}$	$\frac{122}{1309}$
−1	$\frac{10}{1109}$	$\frac{1120}{1129}$ ▲	$\frac{1}{569}$ ▲	$\frac{22}{1609}$	$\frac{1283}{1309}$ ▲
0	$\frac{1}{1109}$	$\frac{11}{1129}$	$\frac{561}{569}$ ▲	$\frac{4}{1609}$	$\frac{27}{1309}$
1	$\frac{111}{1109}$	$\frac{112}{1129}$	$\frac{64}{569}$	$\frac{147}{1609}$	$\frac{123}{1309}$
2	$\frac{122}{1109}$	$\frac{114}{1129}$	$\frac{57}{569}$	$\frac{173}{1609}$	$\frac{124}{1309}$
3	$\frac{234}{1109}$	$\frac{237}{1129}$	$\frac{113}{569}$	$\frac{324}{1609}$	$\frac{274}{1309}$

Note▲: 1 is added to the rightmost digit.

TABLE 7. Fractions whose repetends end with successive terms of $R_{mn \pm p}$, occurring in repeating blocks of one digit

p \ n	1	2	3	4	5
−3	$\frac{499}{1109}$	$\frac{539}{1129}$	$\frac{363}{569}$	$\frac{601}{1609}$	$\frac{583}{1309}$
−2	$\frac{1048}{1109}$ ▲	$\frac{972}{1129}$ ▲	$\frac{510}{569}$ ▲	$\frac{1572}{1609}$ ▲	$\frac{1056}{1309}$ ▲
−1	$\frac{992}{1109}$ ▲	$\frac{1070}{1129}$ ▲	$\frac{472}{569}$ ▲	$\frac{1456}{1609}$ ▲	$\frac{11}{1309}$
0	$\frac{321}{1109}$	$\frac{323}{1129}$	$\frac{207}{569}$	$\frac{411}{1609}$	$\frac{341}{1309}$
1	$\frac{143}{1109}$	$\frac{107}{1129}$	$\frac{51}{569}$	$\frac{221}{1609}$	$\frac{93}{1309}$
2	$\frac{347}{1109}$	$\frac{371}{1129}$	$\frac{161}{569}$	$\frac{479}{1609}$	$\frac{451}{1309}$
3	$\frac{811}{1109}$	$\frac{801}{1129}$	$\frac{419}{569}$	$\frac{1111}{1609}$	$\frac{891}{1309}$

Note ▲: 1 is added to the rightmost digit.

TABLE 8. Fractions whose repetends end with successive terms of $S_{mn \pm p}$, occurring in repeating blocks of one digit

p \ n	1	2	3	4	5
−3	$\frac{409}{1109}$	$\frac{420}{1129}$	$\frac{293}{569}$	$\frac{502}{1609}$	$\frac{698}{1309}$
−2	$\frac{1039}{1109}$	$\frac{992}{1129}$ ▲	$\frac{501}{569}$ ▲	$\frac{1554}{1609}$ ▲	$\frac{1109}{1309}$ ▲
−1	$\frac{1102}{1109}$ ▲	$\frac{42}{1129}$	$\frac{544}{569}$ ▲	$\frac{990}{1609}$ ▲	$\frac{107}{1309}$
0	$\frac{332}{1109}$	$\frac{325}{1129}$	$\frac{200}{569}$	$\frac{437}{1609}$	$\frac{342}{1309}$
1	$\frac{255}{1109}$	$\frac{230}{1129}$	$\frac{107}{569}$	$\frac{372}{1609}$	$\frac{249}{1309}$
2	$\frac{580}{1109}$	$\frac{597}{1129}$	$\frac{282}{569}$	$\frac{799}{1609}$	$\frac{1117}{1309}$
3	$\frac{58}{1109}$ *	$\frac{23}{1129}$ *	$\frac{20}{569}$ *	$\frac{1608}{1609}$	$\frac{1289}{1309}$

Note ▲: 1 is added to the rightmost digit.
*: −1 is added to the rightmost digit.

TABLE 9. Fractions whose repetends end with successive terms of $U_{mn \pm p}$, occurring in repeating blocks of one digit

$p \backslash n$	1	2	3	4	5
−3	$\frac{1019}{1109}$ ▲	$\frac{1010}{1129}$ *	$\frac{499}{569}$ ▲	$\frac{1510}{1609}$ ▲	$\frac{1161}{1309}$ ▲
−2	$\frac{1100}{1109}$ ▲	$\frac{20}{1129}$	$\frac{560}{569}$ ▲	$\frac{1591}{1609}$ ▲	$\frac{53}{1309}$
−1	$\frac{110}{1109}$	$\frac{101}{1129}$	$\frac{72}{569}$	$\frac{1598}{1609}$	$\frac{96}{1309}$
0	$\frac{11}{1109}$	$\frac{2}{1129}$	$\frac{562}{569}$ ▲	$\frac{26}{1609}$	$\frac{1}{1309}$
1	$\frac{112}{1109}$	$\frac{123}{1129}$	$\frac{56}{569}$	$\frac{151}{1609}$	$\frac{150}{1309}$
2	$\frac{233}{1109}$	$\frac{226}{1129}$	$\frac{121}{569}$	$\frac{320}{1609}$	$\frac{247}{1309}$
3	$\frac{356}{1109}$	$\frac{351}{1129}$	$\frac{170}{569}$	$\frac{497}{1609}$	$\frac{398}{1309}$

Note ▲: 1 is added to the rightmost digit.
*: −1 is added to the rightmost digit.

5. Decimal Fractions That Can Be Represented in Terms of Alternating Positive and Negative Tribonacci Series Reading from Right to Left

Starting from Theorem 4 of Johnson [9], we rewrite it as:

The repeating cycle of $\dfrac{(-1)^n \cdot F_p \cdot 10^k - F_{n+p}}{(-1)^n \cdot 10^{2k} - L_n \cdot 10^k + 1}$ ends in F_{mn+p},

and the repeating cycle of $\dfrac{(-1)^n \cdot L_p \cdot 10^k - L_{n+p}}{(-1)^n \cdot 10^{2k} - L_n \cdot 10^k + 1}$ ends in L_{mn+p},

for $m = 1, 2, 3, 4, \ldots$, occurring in blocks of k digits. Substituing (-10^k) for (10^k), we get

Theorem 5:

(15) The repeating cycle of $\dfrac{N}{M} = \dfrac{(-1)^{n+1} \cdot F_p \cdot 10^k - F_{n+p}}{(-1)^n \cdot 10^{2k} + L_n \cdot 10^k + 1}$ ends in F_{mn+p},

(16) and the repeating cycle of $\dfrac{N}{M} = \dfrac{(-1)^{n+1} \cdot L_p \cdot 10^k - L_{n+p}}{(-1)^n \cdot 10^{2k} + L_n \cdot 10^k + 1}$ ends in L_{mn+p},

for $m = 1, 2, 3, 4, \ldots$, occurring in blocks of k digits. If $N/M > 0$, all even terms are negative, if $N/M < 0$, all odd terms are negative. For example,

for $k = 1$, $n = 1$,

$$\begin{aligned} N/M &= 1/89 \\ &= 0 \cdot \ldots 38202247191 \\ &= 0 \cdot \ldots\ldots\ldots \overline{8}5\overline{3}2\overline{1}1 \end{aligned}$$

$$\begin{array}{ll} \ldots 893413\ 5\ 2\ 1 & \text{positive} \\ \hline \ldots.5521\ 8\ 3\ 1 & \text{negative} \\ \hline \ldots.38202247191 & \text{summation} \end{array}$$

for $k = 1$, $n = 12$,

$$N/M = -16/369$$
$$= -0.\overline{04336}$$

$$\begin{array}{rl}
\ldots\ldots 40 & \\
\ldots 323072 & \\
\ldots 11879264 & \\
4807526976 & \\
46368 & \text{positive} \\
\hline
144 & \text{negative} \\
14930352 & \\
\ldots 008755920 & \\
\ldots 6367088 & \\
\ldots\ldots 096 & \\
\ldots\ldots 4 & \\
\hline
\ldots\ldots 60433604336 & \text{summation}
\end{array}$$

Using (15) and (16), we can derive Tables 10 and 11 for $k = 1, 2, 3$, and n from 1 to 7.

TABLE 10. Fractions whose repetends end in F_{nm} with positive and negative terms alternated, positive fractions begin with positive F_{mn}, negative fractions opposite

k \ n	1	2	3	4	5	6	7
1	$\frac{1}{89}$	$\frac{-1}{131}$	$\frac{2}{59}$	$\frac{-3}{171}$	$\frac{-5}{11}$	$\frac{-8}{281}$	$\frac{-13}{191}$
2	$\frac{1}{9899}$	$\frac{-1}{10301}$	$\frac{2}{9599}$	$\frac{-3}{10701}$	$\frac{5}{8899}$	$\frac{-8}{11801}$	$\frac{13}{7099}$
3	$\frac{1}{998999}$	$\frac{-1}{1003001}$	$\frac{2}{995999}$	$\frac{-3}{1007001}$	$\frac{5}{988999}$	$\frac{-8}{1018001}$	$\frac{13}{970999}$

TABLE 11. Fractions whose repetends end in L_{nm} with positive and negative terms alternated, positive fractions begin with positive L_{mn}, negative fractions opposite

k \ n	1	2	3	4	5	6	7
1	$\frac{-19}{89}$	$\frac{-23}{131}$	$\frac{-16}{59}$	$\frac{-27}{171}$	$\frac{9}{11}$	$\frac{-38}{281}$	$\frac{-9}{191}$
2	$\frac{-199}{9899}$	$\frac{-203}{10301}$	$\frac{-196}{9599}$	$\frac{-207}{10701}$	$\frac{-189}{8899}$	$\frac{-218}{11801}$	$\frac{-171}{7099}$
3	$\frac{-1999}{998999}$	$\frac{-2003}{1003001}$	$\frac{-1996}{995999}$	$\frac{-2007}{1007001}$	$\frac{-1980}{988999}$	$\frac{-2018}{1018001}$	$\frac{-1991}{970999}$

Because

$$0.\ \ldots\overline{F}_7F_6\overline{F}_5F_4\overline{F}_3F_2\overline{F}_1 + 0.\ \ldots F_7\overline{F}_6F_5\overline{F}_4F_3\overline{F}_2F_1$$
$$-0.\ 0000\ldots0001 = 0.\ 9999\ldots999,$$
$$0.\ \ldots\overline{F}_7F_6\overline{F}_5F_4\overline{F}_3F_2\overline{F}_1 \quad \text{and} \quad 0.\ \ldots F_7\overline{F}_6F_5\overline{F}_4F_3\overline{F}_2F_1$$

are complementary numbers. This result can be described in another way:

If $N/M > 0$, N/M ends in $0.\ldots F_7\overline{F}_6F_5\overline{F}_4F_3\overline{F}_2F_1$

then $N/M - 1$ ends in $0.\ldots \overline{F}_7F_6\overline{F}_5F_4\overline{F}_3F_2\overline{F}_1$;*

if $N/M < 0$, N/M ends in $0.\ldots \overline{F}_7F_6\overline{F}_5F_4\overline{F}_3F_2\overline{F}_1$

then $1 + N/M$ ends in $0.\ldots F_7\overline{F}_6F_5\overline{F}_4F_3\overline{F}_2F_1$.*

*: −1 is added to the rightmost digit.

So, Tables 10 and 11 have their complementary tables.

From Theorem 4, if we use (-10^k) instead of (10^k), then we will have

Theorem 6: For $N/M > 0$, the repeating cycle of

$$\frac{N}{M} = \frac{T_p \cdot 10^{2k} - (T_{2n+p} - R_nT_{n+p}) \cdot 10^k + T_{n+p}}{-10^{3k} - R_{-n} \cdot 10^{2k} - R_n \cdot 10^k - 1} \tag{17}$$

ends with T_{nm+p}, even terms are negative; for $N/M < 0$, the repeating cycle of

$$\frac{N}{M} = \frac{T_p \cdot 10^{2k} - (T_{2n+p} - R_nT_{n+p}) \cdot 10^k + T_{n+p}}{-10^{3k} - R_{-n} \cdot 10^{2k} - R_n \cdot 10^k - 1} \tag{18}$$

ends with T_{nm+p}, odd terms are negative, both appearing in blocks of k digits. Table 12 shows some illustrations of (17) and (18).

As before, the above results can be developed as follows:

If $N/M > 0$, N/M ends in $0.\ldots T_{7n}\overline{T}_{6n}T_{5n}\overline{T}_{4n}T_{3n}\overline{T}_{2n}T_n$,

then $N/M - 1$ ends in $0.\ldots \overline{T}_{7n}T_{6n}\overline{T}_{5n}T_{4n}\overline{T}_{3n}T_{2n}\overline{T}_n$;*

if $N/M < 0$, N/M ends in $0.\ldots \overline{T}_{7n}T_{6n}\overline{T}_{5n}T_{4n}\overline{T}_{3n}T_{2n}\overline{T}_n$,

then $1 + N/M$ ends in $0.\ldots T_{7n}\overline{T}_{6n}T_{5n}\overline{T}_{4n}T_{3n}\overline{T}_{2n}T_n$.*

*: −1 is added to the rightmost digit.

So, Table 12 has its complementary table, too.

TABLE 12. Fractions whose repetends end in T_{nm} appearing with positive and negative terms alternated

$k \backslash n$	1	2	3	4	5	6	7
1	$\frac{-1}{911}$	$\frac{9}{931}$	$\frac{-12}{1571}$	$\frac{-4}{611}$	$\frac{13}{1111}$	$\frac{-43}{2491}$	$\frac{-14}{211}$
2	$\frac{-1}{990101}$	$\frac{99}{990301}$	$\frac{-102}{1050701}$	$\frac{-4}{951101}$	$\frac{193}{992101}$	$\frac{-313}{1113901}$	$\frac{76}{857101}$
3	$\frac{-1}{999001001}$	$\frac{999}{999003001}$	$\frac{-1002}{1005007001}$	$\frac{-4}{995011001}$	$\frac{1993}{999021001}$	$\frac{-3013}{1011039001}$	$\frac{976}{985071001}$

6. Conclusion

Tables 1-5 and Tables 6-12 have a great difference, the former tables contain blanks, the latter do not. Examining $M = 10^{3k} - R_{-n} \cdot 10^{2k} + R_n \cdot 10^k - 1$, R_n is always greater then R_{-n}, so we can calculate M whenever we wish.

From the above discussion, we can find the following interesting results:

$1/89 = 0.0112358\ldots = 0.\ F_0F_1F_2F_3F_4F_5F_6\ldots,$

$10/89 = 0.112358\ldots = 0.\ F_1F_2F_3F_4F_5F_6\ldots,$

$10/109 = 0.1\bar{1}2\bar{3}5\bar{8}\ldots = 0.\ F_1\bar{F}_2F_3\bar{F}_4F_5\bar{F}_6\ldots,$

$1/109$ ends in $\ldots 853211$ or $\ldots F_6F_5F_4F_3F_2F_1,$

$1/89$ ends in $\ldots \bar{8}5\bar{3}2\bar{1}1$ or $\ldots \bar{F}_6F_5\bar{F}_4F_3\bar{F}_2F_1,$

$88/89$ ends in $\ldots 8\bar{5}3\bar{2}1\bar{1}$ or $\ldots F_6\bar{F}_5F_4\bar{F}_3F_2\bar{F}_1,$*

$100/889 = 0.112485939\ldots = 0.\ T_1T_2T_3T_4T_5T_6T_7\ldots,$

$100/1091 = 0.1\bar{1}2\bar{4}7\ldots = 0.\ T_1\bar{T}_2T_3\bar{T}_4T_5\bar{T}_6T_7\ldots,$

$1/1109$ ends in $\ldots 374211$ or $\ldots T_7T_6T_5T_4T_3T_2T_1,$

$1/911$ ends in $\ldots 3\bar{7}4\bar{2}1\bar{1}$ or $\ldots \bar{T}_7T_6\bar{T}_5T_4\bar{T}_3T_2\bar{T}_1,$

$910/911$ ends in $\ldots \bar{3}7\bar{4}2\bar{1}1$ or $\ldots T_7\bar{T}_6T_5\bar{T}_4T_3\bar{T}_2T_1,$*

*: −1 is added to the rightmost digit.

One of the above,

$1/1109 = 0.00\ldots 862385374211,$

can not only end in T_m, $m = 1, 2, 3, 4, 5, \ldots$, but can also end in T_{9m}, $m = 1, 2, 3, 4, 5, \ldots$. Summing up, we may find different forms of the decimal expansion for a particular fraction. Perhaps, they could be explored on another occasion.

In another article written by this author (unpublished), even Tetrabonacci series can also be divided into four types, as above.

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References

1. Fenton Stancliff. "A Curious Property of α_{ii}." *Scripta Mathematica 19* (1953):126.
2. Calvin T. Long. "The Decimal Expansion of 1/89 and Related Results." *Fibonacci Quarterly 19.1* (1981):53-55.
3. Richard H. Hudson & C. F. Winans. "A Complete Characterization of the Decimal Fractions That Can Be Represented as $\sum 10^{-k(i+1)}F_{\alpha i}$, where $F_{\alpha i}$ is the αi^{th} Fibonacci Number." *Fibonacci Quarterly 19.4* (1981):414-21.
4. C. F. Winans. "The Fibonacci Series in the Decimal Equivalents of Fractions." *A Collection of Manuscripts Related to the Fibonacci Sequence: 18th Anniversary Volume*, ed. Verner E. Hoggatt, Jr. & Marjorie Bicknell-Johnson. Santa Clara, California: The Fibonacci Association, 1980.
5. Pin-Yen Lin. "The General Solution of the Decimal Fraction of Fibonacci Series." *Fibonacci Quarterly 22.3* (1984):229-34.
6. Günter Köhler. "Generating Functions of Fibonacci-Like Sequences and Decimal Expansions of Some Fractions." *Fibonacci Quarterly 23.1* (1985):29-35.
7. Richard H. Hudson. "Convergence of Tribonacci Decimal Expansions." *Fibonacci Quarterly 25.2* (1987):163-70.

8. Pin-Yen Lin. "De Moivre-Type Identities for the Tribonacci Numbers." *Fibonacci Quarterly* *26.2* (1988):131-34.
9. Marjorie Bicknell-Johnson. "Retrograde Renegades and the Pascal Connection: Repeating Decimals Represented by Fibonacci and Other Sequences Appearing from Right to Left." *Fibonacci Quarterly* *27.5* (1989):448-57.
10. Samuel Yates. *Prime Period Length.* Privately published in 1975 by Samuel Yates, 157 Capri D. Kings Point, Delray Beach, FL 33445.
11. Samuel Yates. *Repunits and Repetends.* Boynton Beach, Florida: Star Publishing, 1982.

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BOOK REVIEW

A New Chapter for Pythagorean Triples by A. G. Schaake and J. C. Turner

In this book, the authors develop a new method for generating all Pythagorean triples. They also illustrate that their new method can be used to find solutions to the Pellian equations $x^2 - Ny^2 = \pm 1$ where N is square-free. Since the book contains only accusations and examples, it is impossible to verify that their method is mathematically correct even though the numerous examples found in the book seem to imply that it does work. The authors have published a Departmental Research Report, with proofs of their methods, which may be had, on request, with the book. The reviewer has not read the Research Report.

The method, at least to this reviewer, appears to be new. Furthermore, the method is *a very neat way of relating Pythagorean triples to continued fractions* via what is called a "decision tree." However, the reviewer does not accept the new method with the enthusiasm of the authors because they make claims which, in the opinion of the reviewer, may not be true. Several of these claims will be discussed later in this report.

The basic claim of the authors is essentially that (x, y, z) is a Pythagorean triple if and only if

$$x = \frac{q - r}{2n}, \quad y = \frac{p + s}{2n}, \quad z = \frac{q + r}{2n}$$

where r/s and p/q are, respectively, the last two convergents of a continued fraction of the form

$$[0; u_1, u_2, \ldots, u_i, v, 1, j, (v + 1), u_i, \ldots, u_2, u_1].$$

Using the parity of v, a nice contraction method developed by the authors for the set of values $u_1, u_2, \ldots, u_i$ and the size of j, the authors illustrate that there are five families which predict the value of n.

Most of the book is spent on the development of the techniques used and examples which show how the techniques work. The explanations are clear and the examples are well done. Actually, there are far more examples than are probably needed. The book is very easy to read. In fact, several chapters could be reduced in size or eliminated since anyone with a background in number theory would know most if not all of the material in Chapters 1 and 2. Other parts of the book could also be left out. For example, the tables on pages 127 to 137 were of no value to the reviewer. To be fair to the authors on this point, however, in the Foreword they do state that the material is intended to be accessible to teachers and college students, as well as to number theorists and professional mathematicians. *(Please turn to page 155)*