ON CERTAIN DIVISIBILITY SEQUENCES

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(Submitted June 1988)

In [1], U_n is defined to be a divisibility sequence if $U_m|U_n$ whenever $m|n$. It is conjectured that

$$U_n = A^n \sum_{i=0}^{k} c_i n^i,$$

A, c_i integers, is a divisibility sequence if and only if exactly k of the c_i are 0. In this note, the conjecture will be shown to be true.

Since the A^n factor offers no difficulty, it will be ignored. Furthermore, the sufficiency can be demonstrated easily; therefore, only the necessity will be proven in the following theorem.

Theorem: Let

$$U_n = \sum_{i=0}^{k} c_i n^i,$$

where the c_i are integers and $c_k \neq 0$, be a divisibility sequence; then, $c_i = 0$ for $0 \leq i \leq k - 1$. (Note that there is no loss of generality in assuming that U_n has this form.)

Proof: Let $n = mt$, n, m, t positive integers. Then,

$$U_n = U_{mt} = \sum_{i=0}^{k} c_i (mt)^i = \sum_{i=0}^{k} c_i m^i t^i = \left(\sum_{i=0}^{k} c_i m^i\right) t^k - \sum_{i=0}^{k-1} c^i (t^k - t^i) m^i.$$

Since $U_m|U_n$ for all t, U_m must divide the second sum on the right-hand side. (Note that the first sum is U_m.)

Now, fix $t > 1$ and let $d_i = c_i(t^k - t^i)$ for $0 \leq i \leq k - 1$; note that $t^k - t^i \neq 0$ for all i. Thus,

$$U_m \left| \sum_{i=0}^{k-1} d_i m^i \right. \text{ for all } m.$$

However, U_m is a polynomial in m of degree $k (c_k \neq 0)$; thus, for sufficiently large m,

$$|U_m| > \left| \sum_{i=0}^{k-1} d_i m^i \right|.$$

Hence,

$$\sum_{i=0}^{k-1} d_i m^i = 0 \text{ for all } m.$$

This implies that $d_i = 0$ for all i, and, consequently, $c_i = 0$, $0 \leq i \leq k - 1$.

Reference

1. R. B. McNeill. "On Certain Divisibility Sequences." *Fibonacci Quarterly* *26.2* (1988):169-71.

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