NOTE ON THE RESISTANCE THROUGH A STATIC CARRY LOOK-AHEAD GATE

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(Submitted March 1988)

In this paper, I show that a problem arising in hardware design has a solution that is the ratio of consecutive Fibonacci numbers.

One of the problems in VLSI designs of adders is to minimize the amount of time needed for addition [1]. A straightforward way of adding is to have a separate adder cell for each bit of the operands. The function to be performed by each one-bit adder cell is to take inputs A_i and B_i and a carry bit C_{i-1} from the previous stage, and compute

$$\begin{aligned}\text{SUM}_i &= A_iB_iC_{i-1} + A_i\overline{B}_i\overline{C}_{i-1} + \overline{A}_i\overline{B}_iC_{i-1} + \overline{A}_iB_i\overline{C}_{i-1}\\ &= A_i \oplus B_i \oplus C_{i-1}\end{aligned}$$

and

$$C_i = A_iB_i + A_iC_{i-1} + B_iC_{i-1},$$

where SUM_i is the i^{th} bit of the sum and C_i becomes the carry input to the next stage. Unfortunately, this scheme means that the i^{th} adder cannot compute its result until the $(i-1)^{\text{th}}$ adder has propagated its carry to it.

One way to get around this problem is to look ahead to compute the carry bit to be propagated to each stage. The idea is that each adder can make a quick decision whether to propagate or generate a carry by using the formulas:

$$\text{GEN} = A_iB_i \quad \text{and} \quad \text{PROP} = A_i \oplus B_i.$$

A carry from the previous stage will be propagated if either A_i or B_i is true, and one will be generated at this stage, regardless of the previous carry value, if both A_i and B_i are true. The pull-down transistor part of a 4-stage static carry look-ahead gate as it might be implemented in CMOS or nMOS is shown in Figure 1, where the output is the negation of the fourth carry bit value, the inputs on the left are the zeroth carry bit and the first four PROP values, and the inputs on the right are the first four GEN values.

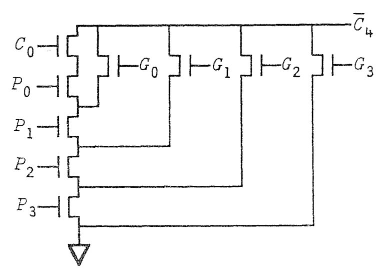

FIGURE 1. 4-stage static carry look-ahead gate

The circuit works by setting things so that the output $\overline{C}_4$ will be high (true) unless there is a path between it and ground. The overbar indicates a negated signal, that is, one which is true when it is at ground and false when it is at the power supply voltage. The transistors can be viewed as switches which allow current to flow if their inputs are high (true). In this circuit, there will be a path to ground if G_3 is true, which means that the fourth stage would generate a carry. If there is no carry generated in the fourth stage, the output can still be pulled low (true) if a carry was propagated through the fourth stage (P_3 is true) and a carry was somehow passed through the third stage. This analysis proceeds recursively, so that if, for example, all the generate bits were false, a carry would only be generated if all the propagate bits were true and the initial C_0 carry was true.

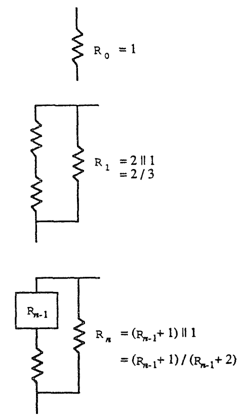

FIGURE 2. Source of the recurrence relation for resistance

In order to compute the delay through this circuit, it is necessary to compute the resistance and capacitance between ground and the output. This note concentrates on the resistance. The approximation made in computing resistance in this paper is that each transistor with a high input is in the conducting state and represents a unit of resistance. A generalized n-stage resistance network for this circuit has a very regular structure, as shown in Figure 2. A "zero-stage" look-ahead gate would comprise but a single resistor and thus have a resistance of one. A one-stage gate has a series of two resistors in parallel with a third; the composite resistance is computed by using the parallel resistance formula:

$$a \| b = \frac{ab}{a + b}.$$

In this case, $a = 2$, since resistors in series sum, and $b = 1$. Thus, $R_1 = 2/3$, and we get a general recurrence relation for R_n:

$$R_0 = 1,$$
$$R_n = \frac{R_{n-1} + 1}{R_{n-1} + 2}.$$

We can attack this recurrence by splitting R_n into its numerator and denominator:

$$R_n = \frac{N_n}{D_n} = \frac{N_{n-1}/D_{n-1} + 1}{N_{n-1}/D_{n-1} + 2} = \frac{N_{n-1} + D_{n-1}}{N_{n-1} + 2D_{n-1}}.$$

So we have a double recurrence:

$$N_0 = 1$$

$$N_n = N_{n-1} + D_{n-1}, \qquad n \geq 1$$

$$D_0 = 1$$

$$D_n = N_{n-1} + 2D_{n-1}. \qquad n \geq 1$$

So far, we have only demonstrated this as a formal solution because the fraction N_n/D_n may not be in lowest terms. The lemma below demonstrates that this is the actual lowest-term solution.

Lemma: N_n and D_n are relatively prime.

Proof: This is a proof by induction. This base case is easy:

$$\gcd(N_0, D_0) = \gcd(1, 1) = 1.$$

Assume that $\gcd(D_{n-1}, N_{n-1}) = 1$. We use a result by Euclid that if $n > m$, then $\gcd(n, m) = \gcd(m, n - m)$ (see [2]). Thus,

$$\begin{aligned}\gcd(D_n, N_n) &= \gcd(N_{n-1} + 2D_{n-1}, N_{n-1} + D_{n-1})\\ &= \gcd(N_{n-1} + D_{n-1}, D_{n-1})\\ &= \gcd(D_{n-1}, N_{n-1})\\ &= 1. \quad \square\end{aligned}$$

We can create ordinary generating functions $N(z)$ and $D(z)$ to find the closed-form solutions for the series. If we define $N_n = D_n = 0$ for $n < 0$ (the ratio R_n will thus be undefined in those cases), then we have formulas for them which are valid for all n:

$$N_n = N_{n-1} + D_{n-1} + \delta_{n0}$$

$$D_n = N_{n-1} + 2D_{n-1} + \delta_{n0}.$$

Multiplying both sides of these equations by z^n and summing over all n gives us the ordinary generating functions:

(1) $\quad N(z) = zN(z) + zD(z) + 1$

(2) $\quad D(z) = zN(z) + 2zD(z) + 1.$

Subtracting (2) - (1) and leaving off the (z)'s for clarity,

$$D - N = zD,$$

or

(3) $\quad N = D(1 - z).$

Plugging this back into (2) gives

$$D = \frac{1}{1 - 3z + z^2}.$$

Hence, by (3),

$$N = \frac{1 - z}{1 - 3z + z^2}.$$

We can get a closed-form expression for N_n from the generating function by factoring the denominator $(1 - 3z + z^2)$ into $(1 - az)(1 - bz)$ and expanding in terms of partial fractions. Using the quadratic formula, we get that

$$a = \frac{3 + \sqrt{5}}{2}, \quad b = \frac{3 - \sqrt{5}}{2}.$$

Here we make the observation that, if we let

$$\phi = \frac{1 + \sqrt{5}}{2}, \quad \hat{\phi} = \frac{1 - \sqrt{5}}{2},$$

then

$$a = \phi^2, \qquad b = \hat{\phi}^2.$$

We can also note that

(4) $\phi^2 - 1 = \phi, \quad \hat{\phi}^2 - 1 = \hat{\phi}$

and

(5) $\phi^2 - \hat{\phi}^2 = \sqrt{5}.$

Therefore, to expand the partial fraction

$$\frac{1 - z}{(1 - \phi^2 z)(1 - \hat{\phi}^2 z)} = \frac{\alpha}{1 - \phi^2 z} + \frac{\beta}{1 - \hat{\phi}^2 z},$$

we can find α by multiplying by $(1 - \phi^2 z)$ and setting z to $1/\phi^2$:

$$\alpha = \frac{\phi}{\sqrt{5}},$$

using identities (4) and (5).

Similarly,

$$\beta = -\frac{\hat{\phi}}{\sqrt{5}}.$$

This gives us a closed form for N_n:

$$N = \sum_n N_n z^n = \alpha \sum_n (\phi^2 z)^n + \beta \sum_n (\hat{\phi}^2 z)^n$$

by substituting the series for the partial fraction form. Equating coefficients of z^n:

$$N_n = \frac{1}{\sqrt{5}}(\phi\phi^{2n} - \hat{\phi}\hat{\phi}^{2n}) = \frac{1}{\sqrt{5}}(\phi^{2n+1} - \hat{\phi}^{2n+1}).$$

We can get D_n from N_n:

$$D_n = N_{n+1} - N_n = \frac{1}{\sqrt{5}}(\phi^{2n+3} - \hat{\phi}^{2n+1}) - \frac{1}{\sqrt{5}}(\phi^{2n+3} - \hat{\phi}^{2n+1})$$
$$= \frac{1}{\sqrt{5}}(\phi^{2n+2} - \hat{\phi}^{2n+2}) = F_{2n+2},$$

where F_i is the i^{th} Fibonacci number [2]. It seems there should have been an easier way to find the solution. We can rewrite the joint recurrences slightly to yield

$$N_0 = 1$$
$$N_n = N_{n-1} + D_{n-1}, \qquad n \geq 1$$
$$D_0 = 1$$
$$D_n = N_n + D_{n-1}. \qquad n \geq 1$$

Therefore, we can build the following table:

n	N_n	D_n	R_n
0	1	1	1.000000
1	2	3	0.666667
2	5	8	0.625000
3	13	21	0.619048
4	34	55	0.618182
5	89	144	0.618056

In other words, we have the Fibonacci numbers alternating between the N_n's and the D_n's. Thus,

$$R_n = \frac{F_{2n+1}}{F_{2n+2}} = \frac{\phi^{2n+1} - \hat{\phi}^{2n+1}}{\phi^{2n+2} - \hat{\phi}^{2n+2}}.$$

It is also possible to compute the asymptotic resistance, since as $n \to \infty$, $\hat{\phi}^n \to 0$ but $\phi^n \to \infty$. This gives

$$R_\infty = \frac{1}{\phi} = \frac{\sqrt{5} - 1}{2}.$$

The convergence, it can be seen, is quite rapid.

A similar result for the resistance through a ladder network was obtained by Basin [3] and independently by Manuel & Santiago [4]. The resistance of their circuit was also a ratio of consecutive Fibonacci numbers, but with the larger number in the numerator:

$$R_n = \frac{F_{2n}}{F_{2n-1}}.$$

References

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2. D. Knuth. *The Art of Computer Programming: Fundamental Algorithms*, Vol. 1, p. 2. Reading, Mass.: Addison-Wesley, 1973.
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4. George Manuel & Amalia Santiago. "An Unexpected Appearance of the Golden Ratio." *College Math. J. 19.2* (1988):168-170.

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