ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-443 *Proposed by Richard Andre-Jeannin, Sfax, Tunisia*

Let us consider the recurrence

$$w_n = mw_{n-1} + w_{n-2},$$

where $m > 0$ is an integer and U_n, V_n the solutions defined by

$$U_0 = 0,\ U_1 = 1;\ V_0 = 2,\ V_1 = m.$$

Show that, if q is an odd divisor of $m^2 + 1$, then

$$V_q \equiv m \pmod{q}.$$

H-444 *Proposed by H.-J. Seiffert, Berlin, Germany*

Show that, for $n = 0, 1, 2, \ldots,$

$$F_n = \sum_{\substack{k=0 \\ (5,\, n-2k)=1}}^{[n/2]} (-1)^{[(n-2k+2)/5]} \binom{n}{k},$$

where (r, s) denotes the greatest common divisor of r and s and [ ] the greatest integer function.

H-445 *Proposed by Paul S. Bruckman, Edmonds, WA*

Establish the identity:

(1) $$\sum_{n=1}^{\infty} \mu(n)\left(\frac{z^n}{1 - z^{2n}}\right) = z - z^2,\ z \in \mathbb{C},\ |z| < 1, \text{ and } \mu \text{ is the Möbius function.}$$

As special cases of (1), obtain the following identities:

(2) $$\sum_{n=1}^{\infty} \mu(2n)/F_{2ns} = -\beta^{2s}\sqrt{5},\ s = 1, 3, 5, \ldots,\ \beta = \tfrac{1}{2}(1 - \sqrt{5});$$

(3) $$\sum_{n=1}^{\infty} \mu(2n-1)/L_{(2n-1)s} = -\beta^{s},\ s = 1, 3, 5, \ldots;$$

(4) $$\sum_{n=1}^{\infty} \mu(n)/F_{ns} = (\beta^{s} - \beta^{2s})\sqrt{5},\ s = 2, 4, 6, \ldots;$$

(5) $\sum_{n=1}^{\infty}(-1)^{n-1}\mu(n)/F_{ns} = (\beta^s + \beta^{2s})\sqrt{5},\ s = 2, 4, 6, \ldots;$

(6) $\sum_{n=1}^{\infty}(-1)^{n-1}\mu(2n-1)/F_{(2n-1)s} = -\beta^s\sqrt{5},\ s = 1, 3, 5, \ldots;$

(7) $\sum_{n=1}^{\infty}(-1)^{n-1}\mu(2n-1)/L_{(2n-1)s} = \beta^s,\ s = 2, 4, 6, \ldots .$

SOLUTIONS

Rather Compact

H-421 *Proposed by Piero Filipponi, Rome, Italy (Vol. 26, no. 2, May 1988)*

Let the numbers $U_n(m)$ (or merely U_n) be defined by the recurrence relation [1]

$$U_{n+2} = mU_{n+1} + U_n;\ U_0 = 0,\ U_1 = 1,$$

where $m \in N = \{1, 2, \ldots\}$.

Find a compact form for

$$S(k, h, n) = \sum_{j=0}^{n-1} U_{k+jh}U_{k+(n-1-j)h} \quad (k, h, n \in N).$$

Note that, in the particular case $m = 1$, $S(1, 1, n) = F_n^{(1)}$ is the n^{th} term of the Fibonacci first convolution sequence [2].

References

1. M. Bicknell. "A Primer on the Pell Sequence and Related Sequences." *Fibonacci Quarterly 13.4* (1975):345-49.
2. V. E. Hoggatt, Jr. "Convolution Triangles for Generalized Fibonacci Numbers." *Fibonacci Quarterly 8.2* (1970):158-71.

Solution by the proposer

It is known [1] that

(1) $U_n = (\alpha^n - \beta^n)/\Delta$ (Binet form)

where

(2) $$\begin{cases} \Delta = (m^2 + 4)^{1/2}, \\ \alpha = (m + \Delta)/2, \\ \beta = (m - \Delta)/2. \end{cases}$$

Analogously, the numbers $V_n(m)$ (or merely V_n) can be defined as either

(3) $V_{n+2} = mV_{n+1} + V_n;\ V_0 = 2,\ V_1 = m,$

or

(4) $V_n = U_{n-1} + U_{n+1} = \alpha^n + \beta^n$ (Binet form).

The following identities will be used throughout the solution:

(5) $V_{j+k} - (-1)^k V_{j-k} = \Delta^2 U_j U_k;$

(6) $$\begin{cases} U_{-n} = (-1)^{n+1}U_n, \\ V_{-n} = (-1)^n V_n; \end{cases}$$

$$(7)\qquad \sum_{j=0}^{r} x^j V_{sj+t} = \frac{(-1)^{s-1}x^{r+2}V_{sr+t} + x^{r+1}V_{s(r+1)+t} + (-1)^t x V_{s-t} - V_t}{(-1)^{s-1}x^2 + V_s x - 1},$$

where s and t are arbitrary integers and x is an arbitrary quantity subject to the restriction $x \neq \alpha^{-s}, \beta^{-s}$.

Identities (5) and (6) can be readily proven with the aid of (1), (2), and (4). The proof of (7) is slightly more complicated but several approaches are possible. One of these proofs is given in "A Matrix Approach to Certain Identities" by P. Filipponi & A. F. Horadam (*Fibonacci Quarterly 26.2* [1988]:115-26).

Now, from (5), we can write

$$U_{k+jh}U_{k+(n-1-j)h} = [V_{2k+(n-1)h} - (-1)^{k+(n-j-1)h}V_{(2j-n+1)h}]/\Delta^2,$$

whence

$$(8)\qquad S(k, h, n) = \frac{nV_{2k+(n-1)h}}{\Delta^2} - \frac{(-1)^{k+(n-1)h}}{\Delta^2}\sum_{j=0}^{n-1}(-1)^{jh}V_{2hj-(n-1)h}$$

$$= \frac{nV_{2k+(n-1)h}}{\Delta^2} - \frac{(-1)^{k+(n-1)h}}{\Delta^2}X_{h,n}.$$

Using (7), (6), and (5), let us calculate the quantity $X_{h,n}$:

Case 1: h is odd [$x = -1$ in (7)]

$$(9)\qquad X_{n,h} = \frac{2(-1)^{n-1}[V_{h(n+1)} + V_{h(n-1)}]}{V_{2h} + 2} = \frac{2(-1)^{n-1}\Delta^2 U_{hn}U_h}{V_{2h} + 2}.$$

Using (1) and (4), (9) becomes

$$(10)\qquad X_{h,n} = 2(-1)^{n-1}U_{hn}/U_h.$$

Case 2: h is even [$x = 1$ in (7)]

$$(11)\qquad X_{h,n} = \frac{2[V_{h(n+1)} - V_{h(n-1)}]}{V_{2h} - 2} = \frac{2\Delta^2 U_{hn}U_h}{V_{2h} - 2} = 2U_{hn}/U_h.$$

From (8), (9), and (10), we obtain

$$(12)\qquad S(k, h, n) = [nV_{2k+(n-1)h} - 2(-1)^k U_{hn}/U_h]/\Delta^2.$$

The relationship (4) allows us to express $S(k, h, n)$ merely in terms of numbers U_n.

As a particular case, we have

$$(13)\qquad S(1, 1, n) = [nV_{n+1} + 2U_n]/\Delta^2 = [nU_{n+2} + (n+2)U_n]/\Delta^2.$$

Also solved by P. Bruckman, L. Kuipers, H.-J. Seiffert, and N. A. Volodin.

Lotsa Sequences

H-422 *Proposed by Larry Taylor, Rego Park, NY*
(Vol. 26, no. 2, May 1988)

(A1) Generalize the numbers (2, 2, 2, 2, 2, 2, 2) to form a seven-term arithmetic progression of integral multiples of Fibonacci and/or Lucas numbers with common difference F_n.

(A2) Generalize the numbers (1, 1, 1, 1, 1, 1) to form a six-term arithmetic progression of integral multiples of Fibonacci and/or Lucas numbers with common difference F_n.

(A3) Generalize the numbers (4, 4, 4, 4, 4) to form a five-term arithmetic progression of integral multiples of Fibonacci and/or Lucas numbers with common difference $5F_n$.

(A4) Generalize the numbers (3, 3, 3, 3), (3, 3, 3, 3), (3, 3, 3, 3) to form three four-term arithmetic progressions of integral multiples of Fibonacci and/or Lucas numbers with common differences F_n, $5F_n$, F_n.

(B) Generalize the Fibonacci and Lucas numbers in such a way that, if the Fibonacci numbers are replaced by the generalized Fibonacci numbers and the Lucas numbers are replaced by the generalized Lucas numbers, the arithmetic progressions still hold.

Solution by Paul S. Bruckman, Edmonds, WA

We indicate below the solutions to parts (A1)-(A4) of the problem:

(A1) $(-2F_{n-2}, F_{n-3}, 2F_{n-1}, L_n, 2F_{n+1}, F_{n+3}, 2F_{n+2})$;

(A2) $(-L_{n-1}, -F_{n-2}, F_{n-1}, F_{n+1}, F_{n+2}, L_{n+1})$;

(A3) $(-4L_{n-1}, -L_{n-3}, 2L_n, L_{n+3}, 4L_{n+1})$;

(A4) (i) $(3F_{n+1}, L_{n+2}, F_{n+4}, 3F_{n+2})$;

(ii) $(-3L_{n-1}, L_{n-2}, L_{n+2}, 3L_{n+1})$;

(iii) $(-3F_{n-2}, -F_{n-4}, L_{n-2}, 3F_{n-1})$.

First, we verify that the above yield the desired solutions:

(A1)
$$\begin{aligned}
-2F_{n-2} + F_n &= -2F_{n-2} + 2F_{n-2} + F_{n-3} = F_{n-3};\\
F_{n-3} + F_n &= F_{n-1} - F_{n-2} + F_{n-1} + F_{n-2} = 2F_{n-1};\\
2F_{n-1} + F_n &= F_{n-1} + F_{n+1} = L_n;\\
L_n + F_n &= F_{n-1} + F_n + F_{n+1} = 2F_{n+1};\\
2F_{n+1} + F_n &= F_{n+1} + F_{n+2} = F_{n+3};\\
F_{n+3} + F_n &= F_{n+2} + F_{n+1} + F_{n+2} - F_{n+1} = 2F_{n+2}. \quad \text{Q.E.D.}
\end{aligned}$$

(A2)
$$\begin{aligned}
&-L_{n-1} + F_n = -F_{n-2} - F_n + F_n = -F_{n-2}; \quad -F_{n-2} + F_n = F_{n-1};\\
&F_{n-1} + F_n = F_{n+1}; \quad F_{n+1} + F_n = F_{n+2}; \quad F_{n+2} + F_n = L_{n+1}. \quad \text{Q.E.D.}
\end{aligned}$$

(A3)
$$\begin{aligned}
-4L_{n-1} + 5F_n &= -4L_{n-1} + L_{n+1} + L_{n-1} = L_n + L_{n-1} - 3L_{n-1}\\
&= L_{n-1} + L_{n-2} - 2L_{n-1} = L_{n-2} - L_{n-1} = -L_{n-3};\\
-L_{n-3} + 5F_n &= -L_{n-3} + L_{n+1} + L_{n-1} = -L_{n-1} + L_{n-2} + L_n + 2L_{n-1}\\
&= L_{n-2} + L_{n-1} + L_n = 2L_n;\\
2L_n + 5F_n &= 2L_n + L_{n-1} + L_{n+1} = L_n + L_{n-1} + L_n + L_{n+1}\\
&= L_{n+1} + L_{n+2} = L_{n+3};\\
L_{n+3} + 5F_n &= L_{n+3} + L_{n-1} + L_{n+1} = L_{n+2} + L_{n+1} + L_{n+1} - L_n + L_{n+1}\\
&= L_{n+1} + L_n + 3L_{n+1} - L_n = 4L_{n+1}. \quad \text{Q.E.D.}
\end{aligned}$$

(A4) (i)
$$\begin{aligned}
3F_{n+1} + F_n &= 2F_{n+1} + F_{n+2} = F_{n+1} + F_{n+3} = L_{n+2};\\
L_{n+2} + F_n &= F_{n+3} + F_{n+1} + F_n = F_{n+3} + F_{n+2} = F_{n+4};\\
F_{n+4} + F_n &= 2F_{n+2} + F_{n+1} + F_{n+2} - F_{n+1} = 3F_{n+2}. \quad \text{Q.E.D.}
\end{aligned}$$

(ii)
$$\begin{aligned}
-3L_{n-1} + 5F_n &= -3L_{n-1} + L_{n+1} + L_{n-1} = L_n + L_{n-1} - 2L_{n-1}\\
&= L_n - L_{n-1} = L_{n-2};\\
L_{n-2} + 5F_n &= L_{n-2} + L_{n-1} + L_{n+1} = L_n + L_{n+1} = L_{n+2};\\
L_{n+2} + 5F_n &= L_{n+1} + L_n + L_{n+1} + L_{n-1} = 3L_{n+1}. \quad \text{Q.E.D.}
\end{aligned}$$

(iii)
$$\begin{aligned}
-3F_{n-2} + F_n &= -3F_{n-3} - 3F_{n-4} + 2F_{n-2} + F_{n-3}\\
&= 2F_{n-2} - 2F_{n-3} - 3F_{n-4} = -F_{n-4};\\
-F_{n-4} + F_n &= -F_{n-2} + F_{n-3} + F_{n-1} + F_{n-2} = L_{n-2};
\end{aligned}$$

$$L_{n-2} + F_n = F_{n-3} + F_{n-1} + F_n = F_{n-1} - F_{n-2} + F_{n-1} + F_{n-1} + F_{n-2}$$
$$= 3F_{n-1}. \quad \text{Q.E.D.}$$

Although not required, it is informative to show how the preceding progressions were discovered. We illustrate the method for part (A1) of the problem. First, we note that the value 2 can be assumed only by the following seven admissible terms: $(F_{-3}, -2F_{-2}, 2F_{-1}, L_0, 2F_1, 2F_2, F_3)$. If we suppose that these are special cases of the desired terms, not necessarily in proper order, we surmise that the general terms of the desired solution may be formed by adding n to each suffix of the preceding list. If so, the asymptotic values of such terms are as follows, again, not necessarily in proper order:

$$\alpha^n 5^{-1/2} \cdot (\alpha^{-3}, -2\alpha^{-2}, 2\alpha^{-1}, 5^{1/2}, 2\alpha, 2\alpha^2, \alpha^3).$$

The terms in parentheses may be crudely approximated as follows: (.24, -.76, 1.24, 2.24, 3.24, 5.24, 4.24). We now rearrange these last terms in ascending order of magnitude: (-.76, .24, 1.24, 2.24, 3.24, 4.24, 5.24), and note that all the terms are indeed in A.P. We now write down the terms of the first list corresponding to these last terms, as follows: $(-2F_{-2}, F_{-3}, 2F_{-1}, L_0, 2F_1, F_3, 2F_2)$. Finally, we add n to each suffix in this last septet, thereby forming the candidate for the desired general solution; as we have verified, this indeed generates the correct solution.

A similar process yields the solutions of the other parts of the problem, though in parts (A3) and (A4) the process is complicated by the fact that the choice of terms forming an A.P. is not unique; moreover, in (A4), a pair of "red herrings" occur, which cannot be used to form an A.P., but these are readily identifiable as such and may quickly be eliminated from consideration.

(B) The appropriate generalization is readily obtained by using the generalized Fibonacci and Lucas numbers defined as follows, for arbitrary constants r and s:

$$U_n = rF_n + sF_{n-1}, \quad V_n = rL_n + sL_{n-1}, \text{ for all integers } n.$$

It is easy to see that the U_n's and V_n's satisfy the Fibonacci recurrence, but have different initial values, in general. From this, we see that the desired generalization is obtained by replacing F by U and L by V in (A1)-(A4); the differences in each A.P. will then be an appropriate multiple (either 1 or 5) of U_n, rather than of F_n. We illustrate only with case (A4)(i):

$$(3U_{n+1}, V_{n+2}, U_{n+4}, 3U_{n+2})$$
$$= (3(rF_{n+1} + sF_n), (rL_{n+2} + sL_{n+1}), (rF_{n+4} + sF_{n+3}), 3(rF_{n+2} + sF_{n+1}))$$
$$= r(3F_{n+1}, L_{n+2}, F_{n+4}, 3F_{n+2}) + s(3F_n, L_{n+1}, F_{n+3}, 3F_{n+1});$$

from (A4)(i), each quadruplet in parentheses is in A.P., with common difference F_n and F_{n-1}, respectively. Due to linearity, the general terms are also in A.P., with common difference $= rF_n + sF_{n-1} = U_n$. Q.E.D.

Also solved by L. Kuipers and the proposer.

A Golden Result

H-423 *Proposed by Stanley Rabinowitz, Littleton, MA*
(Vol. 26, no. 3, August 1988)

Prove that each root of the equation

$$F_n x^n + F_{n+1}x^{n-1} + F_{n+2}x^{n-2} + \cdots + F_{2n-1}x + F_{2n} = 0$$

has an absolute value near ϕ, the golden ratio.

Solution by Tad White, University of California, Los Angeles, CA

Problem: Show that the zeros of the polynomial $F_n z^n + \cdots + F_{2n}$ lie near the circle $|z| = \alpha$, where α is a positive root of $z^2 - z - 1 = 0$.

Solution: First divide through by F_n to obtain a monic polynomial; we will examine the roots of

$$f_n(z) = z^n + \frac{F_{n+1}}{F_n} z^{n-1} + \cdots + \frac{F_{2n}}{F_n}.$$

The following lemma gives us information about the coefficients of f_n.

Lemma 1: If β is the negative root of $z^2 - z - 1 = 0$, then

$$\left|\frac{F_{n+k}}{F_n} - \alpha^k\right| \le \frac{|\beta^n|F_k}{F_n}, \text{ for all } n, k.$$

Proof: Using Binet's formula for F_n, we can write

$$\sqrt{5}(F_{n+k} - F_n\alpha^k) = (\alpha^{n+k} - \beta^{n+k}) - \alpha^k(\alpha^n - \beta^n) = \beta^n(\beta^k - \alpha^k) = -\sqrt{5}\beta^n F_k;$$

dividing by $\sqrt{5}F_n$ and taking absolute values completes the proof. □

If we define $g_n(z) = z^n + \alpha z^{n-1} + \cdots + \alpha^n$, then Lemma 1 tells us that the coefficients of f_n and g_n are close. To make this precise, we can define a norm on the vector space P_n of complex polynomials of degree $\le n$ via

$$\|f\| = \sum_{k=0}^{n} |a_k| \text{ if } f(z) = \sum_{k=0}^{n} a_k z^k.$$

Then Lemma 1 says that

$$\|f_n - g_n\| \le \beta^n \sum_{k=0}^{n} \frac{F_k}{F_n} < \beta^n \frac{F_{n+2}}{F_n} < 3\beta^n.$$

Since $|\beta| < 1$, this says $\|f_n - g_n\| \to 0$ as $n \to \infty$. Note also that

$$g_n(z) = \frac{z^{n+1} - \alpha^{n+1}}{z - \alpha},$$

so the roots of g_n lie on the circle $|z| = \alpha$. Hence, we need only show that the locations of the zeros of a polynomial vary in some sense continuously with the coefficients. This can be made precise via the following lemma.

Lemma 2: Given a sufficiently small $\varepsilon > 0$ and $f_1 \in P_n$, there exists $\delta > 0$ such that if f_0 is an element of P_n with $\|f_0 - f_1\| < \delta$, there exists a one-to-one correspondence between the roots ζ_i of f_0 and the roots η_i of f_1 such that $|\zeta_i - \eta_i| < \varepsilon$ for each i.

Proof: Let $f_t = (1 - t)f_0 + tf_1$ for $0 \le t \le 1$; note that $f_t \in P_n$ for each t. Since the set of zeros of f_1 is discrete, and since ε is small, f_1 does not vanish in the closed punctured ball of radius ε around η_i. Observe that the evaluation maps $e_z : P_n \to \mathbb{C}$, given by $e_z(f) = f(z)$ are continuous with respect to the norm $\|\cdot\|$, and in fact uniformly continuous if we restrict z to the compact set $|z| < 2$ (which contains all of the roots η_i in its interior). Therefore, we can choose δ such that $\|f_0 - f_1\| < \delta$ implies that f_0 does not vanish on $\partial B(\eta_i, \varepsilon)$, where $B(\eta_i, \varepsilon)$ is the closed ball of radius ε at η_i. Since $\|f_0 - f_1\|$ is a monotonic function of t, we have that no f_t vanishes on $\partial B(\eta_i, \varepsilon)$.

Assume further that ε is small enough that the paths $\partial B(\eta_i, \varepsilon)$ are disjoint; then define the functions

$$Z_i(t) = \frac{1}{2\pi i}\int_{\partial B(\eta_i, \varepsilon)} \frac{f_t'(z)}{f_t(z)} dz;$$

(please turn to page 282)