ON A NEW KIND OF NUMBERS

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1. Introduction

Recently, properties of the sequence $\{L_{2n+1}\}$, where L_{2n+1} is a Lucas number of order $2n + 1$, were studied [1]. In the present paper, we introduce a new class of numbers defined by

$$(1.1) \qquad f(n, k) = (-1)^{n-k}\binom{2n+1}{n-k}L_{2k+1},$$

where n is any nonnegative integer and $0 \le k \le n$.

These numbers have the interesting property (see [1], (1.5)):

$$(1.2) \qquad \sum_{k=0}^{n} f(n, k) = 1,$$

for every nonnegative integral value of n. Property (1.2) is very much analogous to the following property of Stirling numbers of the first kind (see [2], (6), p. 145):

$$(1.3) \qquad \sum_{k=1}^{n} S_n^k = 0.$$

Also, these new numbers generalize the Catalan numbers in a nontrivial way. First, we recall that the Catalan numbers c_n are defined by means of the generating relation ([5], p. 82)

$$(1.4) \qquad \sum_{n=0}^{\infty} c_n t^n = (2t)^{-1}(1 - \sqrt{1 - 4t}),$$

or by the explicit formula ([5], p. 101)

$$(1.5) \qquad c_n = (n + 1)^{-1}\binom{2n}{n}.$$

The following relationship is obvious:

$$(1.6) \qquad c_n = \frac{(-1)^n f(n, 0)}{(2n + 1)}.$$

Results obtained in this paper include a table, recurrence relations, generating functions, and summation formulas for these new numbers. In view of (1.6), many results reduce to their corresponding results for the Catalan numbers found in the literature. In our last section, we pose two significant open problems.

As usual $(\alpha)_n$ is Pochhammer's symbol and is defined by

$$(1.7) \qquad (\alpha)_n = \begin{cases} 1 & \text{if } n = 0 \\ \alpha(\alpha + 1) \ldots (\alpha + n - 1), & \text{for all } n \in \{1, 2, 3, \ldots\}, \end{cases}$$

${}_2F_1$ will denote the hypergeometric function defined by

$$(1.8) \qquad {}_2F_1\left[\begin{matrix} a, \; b; \; x \\ c; \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n}\frac{x^n}{n!}, \; c \ne 0, -1, -2, \ldots,$$

and the Jacobi polynomials are defined by

$$(1.9) \qquad P_n^{(\alpha,\beta)}(x) = \frac{(1+\beta)_n}{n!}\left(\frac{x-1}{2}\right)^n {}_2F_1\left[\begin{matrix}-n, & -\alpha - n; \\ 1+\beta; & \end{matrix}\ \frac{x+1}{x-1}\right].$$

2. Table of $f(n, k)$

In this section, we give a table of $f(n, k)$ produced by SCRATCHPAD—IBM's symbolic manipulation language.

TABLE OF $f(n, k)$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	-3	4									
2	10	-20	11								
3	-35	84	-77	29							
4	126	-336	396	-261	76						
5	-462	1320	-1815	1595	-836	199					
6	1716	-5148	7865	-8294	5928	-2587	521				
7	-6435	20020	-33033	39585	-34580	20895	-7815	1364			
8	24310	-77792	136136	-179452	180880	-135320	70856	-23188	3571		
9	-92378	302328	-554268	786828	-883728	771324	-504849	233244	-67849	9349	
10	352716	-1175720	2238390	-3372120	4124064	-4049451	3118185	-1814120	749910	-196329	24476

3. Recurrence Relations

In equation (1.3) of [1], it is noted that

$$(3.1) \qquad 3L_{2n+1} - L_{2n-1} = L_{2n+3}, \ n \ge 2.$$

Using (3.1) with n replaced by $n - 1$ and (1.1), we see that

$$(3.2) \qquad (n+k+1)(n+k)f(n, k) + 3(n+k)(n-k+1)f(n, k-1) + (n-k+1)(n-k+2)f(n, k-2) = 0, \ k \ge 2.$$

Furthermore, eliminating L_{2k+1} from

$$f(n+1, k) = (-1)^{n+1-k}\binom{2n+3}{n+1-k}L_{2k+1}$$

and

$$f(n, k) = (-1)^{n-k}\binom{2n+1}{n-k}L_{2k+1},$$

we obtain the formula

$$(3.3) \qquad f(n+1, k) = \frac{-(2n+3)(2n+2)}{(n-k+1)(n+k+2)}f(n, k).$$

Following the method of proof of formula (3.3) and using (1.7), we can obtain its straightforward generalization in the form

$$(3.4) \qquad f(n+m, k) = \frac{(-1)^m(2n+2)_{2m}}{(n-k+1)_m(n+k+2)_m}f(n, k),$$

where m is a nonnegative integer.

4. Generating Relations

We first obtain generating functions for $f(n, k)$ with respect to n. That is, we prove the following theorem.

Theorem 4.1: Let $f(n, k)$ be defined by (1.1). Then

$$(4.1)\qquad \sum_{n=0}^{\infty}\frac{f(n+k, k)}{2(k+n)+1}t^n = \frac{(1+v)^{2k+1}}{2k+1}L_{2k+1}$$

and

$$(4.2)\qquad \sum_{n=0}^{\infty} f(n+k, k)t^n = \frac{(1+v)^{2k+2}}{1-v}L_{2k+1},$$

where v is the function of t defined by

$$(4.3)\qquad v = -t(1+v)^2.$$

Remark: In view of (1.6), (4.1) yields (1.4) while (4.2) yields the following generating relation for the Catalan numbers:

$$(4.4)\qquad \sum_{n=0}^{\infty}(2n+1)c_n t^n = \frac{(1+v)^2}{1-v},\ v = t(1+v)^2.$$

Proof of (4.1): First, multiply both sides of (1.1) by $(2k+1)/[2(k+n)+1]$. Then sum over n from 0 to ∞. Finally, appeal to the well-known identity ([4], p. 348, Prob. 212),

$$(4.5)\qquad \sum_{n=0}^{\infty}\frac{\alpha}{\alpha+\beta n}\binom{\alpha+\beta n}{n}t^n = (1+u)^{\alpha},\ u = t(1+u)^{\beta}$$

to obtain (4.1).

Proof of (4.2): Starting with the definition (1.1) of $f(n, k)$, we have

$$\sum_{n=0}^{\infty} f(n+k, k)t^n = L_{2k+1}\sum_{n=0}^{\infty}\binom{2n+2k+1}{n}(-t)^n = L_{2k+1}\frac{(1+v)^{2k+2}}{1-v},$$

by virtue of the identity (see [4], p. 349, Prob. 216), which is

$$(4.6)\qquad \sum_{n=0}^{\infty}\binom{\alpha+(\beta+1)n}{n}t^n = \frac{(1+u)^{\alpha+1}}{1-\beta u},$$
$$u = t(1+u)^{\beta+1}.$$

Next, we prove the following theorem on generating functions involving double series:

Theorem 4.2: Let $f(n, k)$ be defined by (1.1). Then

$$(4.7)\qquad \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f(n+k, k)t^{n+k} = (1-t)^{-1},\ |t| < 1,$$

and

$$(4.8)\qquad \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{f(n+k, k)}{2(k+n)+1}t^{n+k} = (1+v)F_4\left[1, \frac{1}{2}; \frac{3}{2}, \frac{1}{2}; \frac{-v}{4}, \frac{-5v}{4}\right],$$

where F_4 is Appell's double hypergeometric function of the fourth kind defined by ([6], p. 14), that is,

$$(4.9)\qquad F_4[a, b; c, c'; x, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_m}\frac{x^m}{m!}\frac{y^n}{n!},\ \sqrt{|x|}+\sqrt{|y|} < 1.$$

Proof of (4.7): By making use of (1.1), we observe that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f(n+k, k)t^{n+k} = \sum_{k=0}^{\infty}L_{2k+1}t^k\sum_{n=0}^{\infty}\binom{2n+2k+1}{n}(-t)^n.$$

Summing the inner series with the help of (4.6) and then interpreting the resulting expression by means of the generating relation (1.4) of [1], which is,

$$(4.10)\qquad \sum_{n=0}^{\infty}L_{2n+1}t^n = (1+t)(1-3t+t^2)^{-1},\ |t| < 1,$$

we are led to (4.7). Alternatively, (4.7) can be obtained by using (1.2).

Proof of (4.8): Comparing (4.10) with the following generating function for Jacobi polynomials (see [3], Eq. 10, p. 256),

$$(4.11)\quad \sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n P_n^{(\alpha,\beta)}(x)}{(1+\alpha)_n} t^n = (1-t)^{-1-\alpha-\beta}\, {}_2F_1\left[\begin{matrix}\frac{1}{2}(1+\alpha+\beta),\ \frac{1}{2}(2+\alpha+\beta); & \frac{2t(x-1)}{(1-t)^2}\\ 1+\alpha; & \end{matrix}\right]$$

we obtain the formula

$$(4.12)\quad L_{2n+1} = \frac{n!}{(1/2)_n} P_n^{(1/2,-1/2)}(3/2).$$

If we use (4.12) in the following generating function (see [3], p. 271),

$$(4.13)\quad \sum_{n=0}^{\infty} \frac{(\gamma)_n(\delta)_n P_n^{(\alpha,\beta)}(x)}{(1+\alpha)_n(1+\beta)_n} t^n = F_4[\gamma,\delta;\,1+\alpha,\ 1+\beta;\ \tfrac{1}{2}t(x-1),\ \tfrac{1}{2}t(x+1)],$$

remembering that

$$(4.14)\quad (3/2)_n = (1/2)_n(2n+1),$$

we find that L_{2n+1} satisfies the generating relation

$$(4.15)\quad \sum_{n=0}^{\infty} \frac{L_{2n+1}}{2n+1} t^n = F_4\left(1,\ \frac{1}{2};\ \frac{3}{2},\ \frac{1}{2};\ \frac{t}{4},\ \frac{5t}{4}\right).$$

If we now start with the left-hand side of (4.8), we have

$$\sum_{n,k=0}^{\infty} \frac{f(n+k,\ k)}{2(n+k)+1} t^{n+k} = \sum_{k=0}^{\infty} \frac{L_{2k+1}}{2k+1} t^k \sum_{n=0}^{\infty} \frac{2k+1}{2(k+n)+1}\binom{2n+2k+1}{n}(-t)^n.$$

Summing the inner series by using (4.5), we get

$$\sum_{n,k=0}^{\infty} \frac{f(n+k,\ k)t^{n+k}}{2(n+k)+1} = (1+v)\sum_{k=0}^{\infty} \frac{L_{2k+1}}{2k+1}(-v)^k.$$

Interpreting the last infinite series by means of (4.15) along with the second member of the generating function, (4.8) follows at once. This concludes the proof of Theorem 4.2.

5. Summation Formulas

In this section we propose to prove the following summation formulas:

$$(5.1)\quad \sum_{m=0}^{n-1}\left\{f(n+k-m-1,\ k) + \frac{L_{2k+1}}{L_{2k-1}} f(n+k-m-1,\ k-1)\right\} f(m,\ 0) = f(n+k,\ k) - f(n,\ 0)L_{2n+1},\quad k \ge 1.$$

$$(5.2)\quad \sum_{m=0}^{n-1} \frac{f(m,\ 0)f(n-m-1,\ 0)}{(2m+1)\{2(n-m)-1\}} = -\frac{f(n,\ 0)}{2n+1}.$$

$$(5.3)\quad \sum_{m=0}^{\infty} \frac{\{n-2m(1+k)\}}{(2m+1)\{2(k+n-m)+1\}} f(n+k-m,\ k)f(m,\ 0) = 0.$$

$$(5.4)\quad \sum_{m=0}^{n-1} f(m,\ 0)\left[\frac{f(n-m+k-1,\ k)}{2(k+n-m)-1} + \frac{L_{2k+1}}{L_{2k-1}}\frac{f(n+k-m-1,\ k-1)}{(2m+1)(2k+1)}\right] = \frac{f(n+k,\ k)}{2(k+n)+1} - \frac{L_{2k+1}}{(2k+1)}\frac{f(n,\ 0)}{(2n+1)},\quad k \ge 1.$$

$$(5.5)\quad \sum_{k=0}^{n}(-1)^k\left[\frac{f(n,\ n-k)}{L_{2(n-k)+1}}+\frac{f(n,\ n-k+1)}{L_{2(n-k)+3}}\right]=(2n+1)c_n.$$

$$(5.6)\quad \sum_{k=0}^{n}\frac{f(n,\ k)f(n,\ n-k)}{L_{2k+1}L_{2(n-k)+1}}=\frac{f(2n,\ n)}{L_{2n+1}}-\frac{f(2n,\ n+1)}{L_{2n+3}}.$$

Remark: In view of relationship (1.6), (5.2) and (5.3) yield the following formulas for the Catalan numbers:

$$(5.7)\quad \sum_{m=0}^{n}c_m c_{n-m}=c_{n+1},$$

and

$$(5.8)\quad n\sum_{m=0}^{\infty}c_m c_{n-m}=2\sum_{m=0}^{\infty}mc_m c_{n-m},$$

respectively.

Proof of (5.1): Changing the dummy index k to $k-1$ in (4.2), we get

$$(5.9)\quad \sum_{n=0}^{\infty}f(n+k-1,\ k-1)t^n=L_{2k-1}\frac{(1+v)^{2k}}{1-v}.$$

On the other hand, for $k=0$, (4.2) reduces to

$$(5.10)\quad \sum_{n=0}^{\infty}f(n,\ 0)t^n=\frac{(1+v)^2}{1-v}.$$

In view of (5.9), (4.2) can be written as

$$\sum_{n=0}^{\infty}f(n+k,\ k)t^n=\frac{L_{2k+1}}{L_{2k-1}}(1+v)^2\sum_{n=0}^{\infty}f(n+k-1,\ k-1),$$

which can be rewritten in the form

$$\left[1-t\sum_{m=0}^{\infty}f(m,\ 0)t^m\right]\sum_{n=0}^{\infty}f(n+k,\ k)t^n$$
$$=\frac{L_{2k+1}}{L_{2k-1}}\sum_{m,n=0}^{\infty}f(m,\ 0)f(n+k-1,\ k-1)t^{n+m}$$

by virtue of (5.10).

Equating the coefficients of t^n and using (2.4), we arrive at (5.1).

Proof of (5.2): Setting $k=0$ in (4.1), we get

$$(5.11)\quad \sum_{n=0}^{\infty}\frac{f(n,\ 0)}{2n+1}t^n=1+v.$$

In view of the definition of v in (4.3), (5.11) can be written as

$$\sum_{n=0}^{\infty}\frac{f(n,\ 0)}{2n+1}t^n=1-t(1+v)^2=1-t\sum_{n,m=0}^{\infty}\frac{f(n,\ 0)f(m,\ 0)}{(2n+1)(2m+1)}t^{n+m}.$$

Comparing coefficients of t^n, we get (5.2).

Proof of (5.3): Combining (4.1) and (4.2), we have

$$\sum_{n=0}^{\infty}f(n+k,\ k)t^n=\frac{(2k+1)(1+v)}{1-v}\sum_{n=0}^{\infty}\frac{f(n+k,\ k)}{2(k+n)+1}t^n.$$

Multiplying both sides by $(1+v)$ and then using (5.10) and (5.11), we obtain

$$\sum_{n,m=0}^{\infty}\frac{f(m,\ 0)f(n+k,\ k)}{2m+1}t^{n+m}=(2k+1)\sum_{n,m=0}^{\infty}\frac{f(m,\ 0)f(n+k,\ k)}{2(k+n)+1}t^{n+m}.$$

By equating the coefficients of t^n, we get (5.3).

Proof of (5.4): This proof is similar to that of (5.1) and, hence, will be omitted.

Proof of (5.5); Equation (5.5) is an immediate consequence of (1.1), the following identity (see [5], p. 65),

$$(5.12)\qquad \binom{2n+1}{k} = \sum_{k=0}^{n}\left[\binom{2n+1}{k} - \binom{2n+1}{k-1}\right],$$

and (1.5).

Proof of (5.6): Using (1.1), we have

$$\sum_{k=0}^{n}\frac{f(n,k)f(n,n-k)}{L_{2k+1}L_{2(n-k)+1}} = (-1)^k\sum_{k=0}^{n}\binom{2n+1}{k}\binom{2n+1}{n-k} = (-1)^k\binom{4n+2}{n},$$

where we obtained the last equation by using the Vandermonde addition formula, which is

$$(5.13)\qquad \sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n}.$$

Appealing to the binomial identity

$$(5.14)\qquad \binom{x+1}{n} = \binom{x}{n} + \binom{x}{n-1},$$

we have

$$\sum_{k=0}^{n}\frac{f(n,k)f(n,n-k)}{L_{2k+1}L_{2(n-k)+1}} = (-1)^n\left\{\binom{4n+1}{n} + \binom{4n+1}{n-1}\right\}.$$

Using (1.1) to interpret the right-hand side, we arrive at (5.6).

6. Questions

The most obvious questions arising from this work are:

(i) Do the numbers $f(n, k)$ have a nice combinatorial meaning?

(ii) We have seen that $f(n, k)$ has a property analogous to the Stirling numbers of the first kind and that they also generalize the Catalan numbers. Is it possible to associate $f(n, k)$ with some other known mathematical objects?

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