AN ALGEBRAIC IDENTITY AND SOME PARTIAL CONVOLUTIONS

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Let $\{a_i\}_{i \geq 0}$ and $\{b_i\}_{i \geq 0}$ be any two real or complex number sequences satisfying $a_i \neq 0$ and $b_i \neq 0$ for $i > 0$. Assume that x, y, and z are three formal variables. For any natural number n, define a formal binomial coefficient as follows:

$$(1) \qquad \binom{x}{n}_{(a)} = \prod_{k=1}^{n} \frac{x - a_{k-1}}{a_k}, \text{ where } \binom{x}{0}_{(a)} = 1.$$

It is obvious that when $a_k = k$ $(k = 0, 1, \ldots)$, $\binom{x}{n}_{(a)}$ reduces to the ordinary binomial coefficient. If we replace x by $1 - q^x$ and a_k by $1 - q^k$ $(k = 0, 1, \ldots)$ instead, then $\binom{x}{n}_{(a)}$ becomes the Gaussian binomial coefficient

$$q^{\binom{n}{2}}\begin{bmatrix} x \\ n \end{bmatrix}.$$

Based on these preliminaries, we are ready to state our main result.

Theorem: Let $0 \leq m \leq n \leq r$ be three natural numbers. Then the following algebraic identity holds:

$$(2) \qquad \sum_{k=m}^{n} \{b_{r-k}(x - a_k)z - a_k(y - b_{r-k})\}\binom{x}{k}_{(a)}\binom{y}{r-k}_{(b)} z^k$$
$$= a_{n+1}b_{r-n}\binom{x}{n+1}_{(a)}\binom{y}{r-n}_{(b)} z^{n+1} - a_m b_{r-m+1}\binom{x}{m}_{(a)}\binom{y}{r-m+1}_{(b)} z^m.$$

This identity follows from splitting the summand

$$\{b_{r-k}(x - a_k)z - a_k(y - b_{r-k})\}\binom{x}{k}_{(a)}\binom{y}{r-k}_{(b)} z^k$$
$$= a_{k+1}b_{r-k}\binom{x}{k+1}_{(a)}\binom{y}{r-k}_{(b)} z^{k+1} - a_k b_{r-k+1}\binom{x}{k}_{(a)}\binom{y}{r-k+1}_{(b)} z^k$$

and diagonal cancellation.

Taking $z = 1$, (2) reduces to the following.

Corollary: Let $0 \leq m \leq n \leq r$ be three natural numbers, then

$$(3) \qquad \sum_{k=m}^{n} (b_{r-k}x - a_k y)\binom{x}{k}_{(a)}\binom{y}{r-k}_{(b)}$$
$$= a_{n+1}b_{r-n}\binom{x}{n+1}_{(a)}\binom{y}{r-n}_{(b)} - a_m b_{r-m+1}\binom{x}{m}_{(a)}\binom{y}{r-m+1}_{(b)}.$$

For the remainder of the paper, we shall discuss the applications of (2) and (3) to combinatorial identities.

First, letting $m = 0$, $y = m - x$, and $a_k = b_k = k$ in (3) gives

$$(4) \qquad \sum_{k=0}^{n} \frac{rx - mk}{rx}\binom{x}{k}\binom{m-x}{r-k} = \frac{r-n}{r}\binom{x-1}{n}\binom{m-x}{r-n}.$$

If we define a partial convolution by

$$(5) \qquad S_m(x, r, n) = \sum_{k=0}^{n} \binom{x}{k}\binom{m-x}{r-k},$$

then (4) generates the following recurrence:

$$S_m(x,\ r,\ n) = \frac{r-n}{r}\binom{x-1}{n}\binom{m-x}{r-n} + \frac{m}{r} S_{m-1}(x-1,\ r-1,\ n-1).$$

Performing iteration on this recurrence and noting that the closed form of $S_0(x,\ r,\ n)$ from (4) and (5) is

$$(6)\qquad \sum_{k=0}^{n}\binom{x}{k}\binom{-x}{r-k} = \frac{r-n}{r}\binom{x-1}{n}\binom{-x}{r-n},$$

we have

$$(7)\qquad S_m(x,\ r,\ n) = \sum_{k=0}^{m}\frac{r-n}{r-k}\binom{m}{k}\binom{r}{k}^{-1}\binom{x-k-1}{n-k}\binom{m-x}{r-n},$$

which contains the following interesting example (cf. Anderson [1]):

$$(8)\qquad \sum_{k=0}^{n}\binom{x}{k}\binom{1-x}{r-k} = \frac{(1-x)(r-1)-n}{r(r-1)}\binom{x-1}{n}\binom{-x}{r-n-1}.$$

This identity and (6) are the main results of [1] established by the induction principle.

Rewriting (7) in the form

$$S_m(x,\ r,\ n) = (-1)^{m+n}\frac{r-n}{r}\binom{m-x}{r-n}\binom{r-1}{m}^{-1}\sum_{k=0}^{m}\binom{m-r}{m-k}\binom{n-x}{n-k},$$

and making some trivial modifications, it may be reformulated as

$$(9)\qquad \sum_{i=0}^{n}\binom{n}{i}^{-1}\binom{x}{i}\binom{m-x-r+n}{n-i}\binom{r}{i} = \frac{\binom{n-r}{n}}{\binom{m-r}{m}}\sum_{k=0}^{n}\binom{m-r}{m-k}\binom{n-x}{n-k}.$$

Since (9) is a polynomial identity in r, it is also true if we replace r by a continuous variable y which provides an algebraic identity. The particular case of $m = 0$ in (9) yields the following combinatorial identity:

$$(10)\qquad \sum_{k=0}^{n}\binom{x}{k}\binom{y}{k}\binom{n}{k}^{-1}\binom{n-x-y}{n-k} = \binom{n-x}{n}\binom{n-y}{n}.$$

Next, taking $a_i = b_i = i$ and replacing r and n by $m+n$ in (3), we have

$$(11)\qquad \sum_{k=0}^{n}\{(m+k)y-(n-k)x\}\binom{x}{m+k}\binom{x-1}{n-k} = my\binom{x}{m}\binom{y-1}{n}.$$

Putting $x = y$ and $m = n + 1$ in (11), we obtain the following identity,

$$(12)\qquad \sum_{k=0}^{n}\frac{2k+1}{n+k+1}\binom{x}{n-k}\binom{x-1}{n+k} = \binom{x-1}{n}^2,$$

which reduces to an identity of Riordan ([3], p. 18) for $x = n - m$.

If we let $m \to m + s$, $x \to 2m + s$, and $y \to 2n + s$, alternatively, then (11) degenerates to Prodinger's generalization for Riordan's identity (cf. [2], and [3], p. 89):

$$(13)\qquad \sum_{k\ge 0}(2k+s)\binom{2m+s}{m-k}\binom{2n+s}{n-k} = \frac{(m+s)(n+s)}{m+n+s}\binom{2m+s}{m}\binom{2n+s}{n}.$$

Finally, letting $x = y = 1 - q^t$, $a_i = b_i = 1 - q^i$, and $m = 0$ in (3), we obtain the following q-binomial convolution formula by simple computation:

$$(14)\qquad \sum_{k=0}^{n}\frac{1-q^{r-2k}}{1-q^{r-n}}\begin{bmatrix}t\\k\end{bmatrix}\begin{bmatrix}t\\r-k\end{bmatrix}q^{(n-k)(r-n-k-1)} = \begin{bmatrix}t-1\\n\end{bmatrix}\begin{bmatrix}t\\r-n\end{bmatrix}.$$

When $q \to 1$, (14) reduces to the ordinary binomial identity:

$$(15)\qquad \sum_{k=0}^{n}\frac{r-2k}{r-n}\binom{x}{k}\binom{x}{r-k} = \binom{x-1}{n}\binom{x}{r-n}.$$

References

1. E. S. Anderson. "Two Summation Formulae for Product Sums of Binomial Coefficients." *Math. Scand.* *1* (1953):261-62.
2. H. Prodinger. "The Average Height of a Stack Where Three Operations Allowed and Some Related Problems." *J. Combin., Inform. and System Sciences* 5.4 (1980):287-304.
3. J. Riordan. *Combinatorial Identities*. New York: John Wiliey, 1968.

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