ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. Hillman

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should include solutions.

Anyone desiring acknowledgment of contributions should enclose a stamped, self-addressed card (or envelope).

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n, satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

<u>B-670</u> *Proposed by Russell Euler, Northwest Missouri State U., Marysville, MO*

Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{nF_n}{2^n}$.

<u>B-671</u> *Proposed by Herta T. Freitag, Roanoke, VA*

Show that all even perfect numbers are hexagonal and hence are all triangular. [A perfect number is a positive integer which is the sum of its proper positive integral divisors. The hexagonal numbers are {1, 6, 15, 28, 45, ...} and the triangular numbers are {1, 3, 6, 10, 15, ...}.]

<u>B-672</u> *Proposed by Philip L. Mana, Albuquerque, NM*

Let S consist of all positive integers n such that $n = 10p$ and $n + 1 = 11q$, with p and q primes. What is the largest positive integer d such that every n in S is a term in an arithmetic progression a, $a + d$, $a + 2d$, ...?

<u>B-673</u> *Proposed by Paul S. Bruckman, Edmonds, WA*

Evaluate the infinite product $\displaystyle\prod_{n=2}^{\infty} \frac{F_{2n} + 1}{F_{2n} - 1}$.

<u>B-674</u> *Proposed by Richard André-Jeannin, Sfax, Tunisia*

Define the sequence $\{u_n\}$ by

$$u_0 = 0,\ u_1 = 1,\ u_n = gu_{n-1} - u_{n-2},\ \text{for } n \text{ in } \{2, 3, \ldots\},$$

where g is a root of $x^2 - x - 1 = 0$. Compute u_n for n in {2, 3, 4, 5} and then deduce that $(1 + \sqrt{5})/2 = 2\cos(\pi/5)$ and $(1 - \sqrt{5})/2 = 2\cos(3\pi/5)$.

<u>B-675</u> *Proposed by Richard André-Jeannin, Sfax, Tunisia*

In a manner analogous to that for the previous problem, show that

$$\sqrt{2+\sqrt{2}} = 2\cos\frac{\pi}{8} \quad \text{and} \quad \sqrt{2-\sqrt{2}} = 2\cos\frac{3\pi}{8}.$$

SOLUTIONS

Not True Asymptotically

<u>B-645</u> *Proposed by R. Tošić, U. of Novi Sad, Yugoslavia*

$$\text{Let } G_{2m} = \binom{2m-1}{m} - 2\binom{2m-1}{m-3} + \binom{2m}{m-5} \text{ for } m = 1, 2, 3, \ldots,$$

$$G_{2m+1} = \binom{2m}{m} - \binom{2m+1}{m-2} + 2\binom{2m}{m-5} \qquad \text{for } m = 0, 1, 2, \ldots,$$

where $\binom{n}{k} = 0$ for $k < 0$. Prove or disprove that $G_n = F_n$ for $n = 0, 1, 2, \ldots$.

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY

Let us study the asymptotic growth of G_n. It is evident that

$$G_{2m} \sim \binom{2m-1}{m} \quad \text{and} \quad G_{2m+1} \sim \binom{2m}{m}.$$

Hence,

$$\frac{G_{2m+1}}{G_{2m}} \sim \frac{\binom{2m}{m}}{\binom{2m-1}{m}} = \frac{(2m)!}{m!m!} \cdot \frac{m!(m-1)!}{(2m-1)!} = 2$$

and

$$\frac{G_{2m+2}}{G_{2m+1}} \sim \frac{\binom{2m+1}{m+1}}{\binom{2m}{m}} = \frac{(2m+1)!}{(m+1)!m!} \cdot \frac{m!m!}{(2m)!} = \frac{2m+1}{m+1} \sim 2,$$

so that $G_n/G_{n-1} \sim 2$. However, it is well known that

$$F_n/F_{n-1} \sim (1+\sqrt{5})/2.$$

Thus, $G_n \neq F_n$ for sufficiently large n. In fact, from numerical computations, we have $G_n = F_n$ for $0 < n \leq 14$, and $G_n > F_n$ for $n \geq 15$.

Also solved by Charles Ashbacher, Paul S. Bruckman, James E. Desmond, Piero Filipponi, L. Kuipers, and the proposer.

Triangular Number Analogue

<u>B-646</u> *Proposed by A. P. Hillman in memory of Gloria C. Padilla*

We know that $F_{2n} = F_nL_n = F_n(F_{n-1} + F_{n+1})$. Find m as a function of n so as to have the analogous formula $T_m = T_n(T_{n-1} + T_{n+1})$, where T_n is the triangular number $n(n+1)/2$.

Solution by H.-J. Seiffert, Berlin, Germany

We have:
$$T_n(T_{n-1} + T_{n+1}) = T_n(T_n - n + T_n + n + 1) = T_n(2T_n + 1)$$
$$= n(n+1)(n(n+1)+1)/2 = T_{n(n+1)}.$$

Also solved by Richard André-Jeannin, Wray G. Brady, Paul S. Bruckman, Nicos D. Diamantis, Russell Euler, Piero Filipponi, Herta T. Freitag, Russell Jay Hendel, L. Kuipers, Jack Lee, Carl Libis, Bob Prielipp, Jesse Nemoyer & Joseph J. Kostal & Durbha Subramanyam, Sahib Singh, Lawrence Somer, Gregory Wulczyn, and the proposer.

Much Ado about Zero

B-647 *Proposed by L. Kuipers, Serre, Switzerland*

Simplify

$$[L_{2n} + 7(-1)^n][L_{3n+3} - 2(-1)^n L_n] - 3(-1)^n L_{n-2}L_{n+2}^2 - L_{n-2}L_{n-1}L_{n+2}^3.$$

Solution by Sahib Singh, Clarion U. of Pennsylvania, Clarion, PA

The given expression simplifies to zero. By using the Binet form of Lucas numbers, it follows that $L_{2n} + 7(-1)^n = L_{n-2}L_{n+2}$. In view of this, the given expression is

$$L_{n-2}L_{n+2}[L_{3n+3} - 2(-1)^n L_n - (L_{n+2}^2 L_{n-1} + 3(-1)^n L_{n+2})].$$

Again, applying the Binet form of Lucas numbers, we see that

$$L_{n+2}^2 L_{n-1} + 3(-1)^n L_{n+2} = L_{3n+3} - 2(-1)^n L_n.$$

Hence, the required conclusion follows.

Also solved by Paul S. Bruckman, Herta T. Freitag, Y. H. Harris Kwong, Carl Libis, Bob Prielipp, H.-J. Seiffert, M. Wachtel, Gregory Wulczyn, and the proposer.

Pell Primitive Pythagorean Triples

B-648 *Proposed by M. Wachtel, Zurich, Switzerland*

The Pell numbers P_n and Q_n are defined by

$$P_{n+2} = 2P_{n+1} + P_n,\ P_0 = 0,\ P_1 = 1;\ Q_{n+2} = 2Q_{n+1} + Q_n,\ Q_0 = 1 = Q_1.$$

Show that $(P_{4n}, P_{2n}^2 + 1, 3P_{2n}^2 + 1)$ is a primitive Pythagorean triple for n in $\{1, 2, \ldots\}$.

Solution by Paul S. Bruckman, Edmonds, WA

The Pell numbers satisfy the following identities:

(1) $\quad 2P_{2n}Q_{2n} = P_{4n}$;

(2) $\quad Q_{2n}^2 - 2P_{2n}^2 = 1$.

Hence,

(3) $\quad Q_{2n}^2 - P_{2n}^2 = P_{2n}^2 + 1$.

It is known that primitive Pythagoren triples are generated by

(4) $\quad (2ab, a^2 - b^2, a^2 + b^2)$, where g.c.d. $(a, b) = 1$.

We may let $a = Q_{2n}$, $b = P_{2n}$. We see from (2) that g.c.d. $(a, b) = 1$. Also

$2ab = P_{4n}$ [using (1)];

$a^2 - b^2 = P_{2n}^2 + 1$ [using (3)];

and

$$a^2 + b^2 = 3P_{2n}^2 + 1 \quad \text{[adding } 2b^2 \text{ to both sides of (3)].}$$

This proves the assertion.

Also solved by Nicos D. Diamantis, Ernest J. Eckert, Russell Euler, Piero Filipponi, Herta T. Freitag, Russell Jay Hendel, L. Kuipers, Jesse Nemoyer & Joseph J. Kostal & Durbha Subramanyam, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, Gregory Wulczyn, and the proposer.

Sides Differing by 17

B-649 *Proposed by M. Wachtel, Zurich, Switzerland*

Give a rule for constructing a sequence of primitive Pythagorean triples (a_n, b_n, c_n) whose first few triples are in the table

n	1	2	3	4	5	6	7	8
a_n	24	28	88	224	572	1248	3276	7332
b_n	7	45	105	207	555	1265	3293	7315
c_n	25	53	137	305	797	1777	4645	10357

and which satisfy

$$|a_n - b_n| = 17,$$

$$a_{2n-1} + a_{2n} = 26P_{2n} = b_{2n-1} + b_{2n},$$

and $$c_{2n-1} + c_{2n} = 26Q_{2n}.$$

[P_n and Q_n are the Pell numbers of B-648.]

Rule by Paul S. Bruckman, Edmonds, WA

$$(a_{2n-1}, b_{2n-1}, c_{2n-1})$$
$$= (10P_n^2 + 26P_nQ_n - 12Q_n^2, -24P_n^2 + 26P_nQ_n + 5Q_n^2, 26P_n^2 - 14P_nQ_n + 13Q_n^2),$$

$$(a_{2n}, b_{2n}, c_{2n})$$
$$= (-10P_n^2 + 26P_nQ_n + 12Q_n^2, 24P_n^2 + 26P_nQ_n - 5Q_n^2, 26P_n^2 + 14P_nQ_n + 13Q_n^2).$$

Rule by Ernest J. Eckert, U. of South Carolina, Aiken, SC

Let $(a_1, b_1, c_1) = (24, 7, 25)$, $(a_2, b_2, c_2) = (28, 45, 53)$ and A denote the matrix

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

Then $[a_{2n-1} \quad b_{2n-1} \quad c_{2n-1}]$ is the matrix product $[a_1 \quad b_1 \quad c_1]A^{n-1}$ and

$$[a_{2n} \quad b_{2n} \quad c_{2n}] = [a_2 \quad b_2 \quad c_2]A^{n-1}.$$

Editor's note: The derivations and proofs given by Bruckman and Eckert are not included because of space limitations; however, since each term in the required equations satisfies the same 3rd order linear homogeneous recursion

$$w_{n+3} = 5(w_{n+2} + w_{n+1}) - w_n,$$

it suffices to verify the rules for $n = 1$, 2, and 3.

Also solved by Gregory Wulczyn and the proposer.

Average Age of Generalized Rabbits

B-650 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome Italy & David Singmaster, Polytechnic of the South Bank, London, UK*

Let us introduce a pair of 1-month-old rabbits into an enclosure on the first day of a certain month. At the end of one month, rabbits are mature and each pair produces $k - 1$ pairs of offspring. Thus, at the beginning of the second month there is 1 pair of 2-month-old rabbits and $k - 1$ pairs of 0-month-olds. At the beginning of the third month, there is 1 pair of 3-month-olds, $k - 1$ pairs of 1-month-olds, and $k(k - 1)$ pairs of 0-month-olds. Assuming that the rabbits are immortal, what is their average age A_n at the end of the n^{th} month? Specialize to the first few values of k. What happens as $n \to \infty$?

Solution by Sahib Singh, Clarion U. of Pennsylvania, Clarion, PA

If A_i denotes the average age at the end of the i^{th} month, then we have the following recurrence relation:

$$A_{i+1} = \frac{1}{k}(1 + A_i), \text{ where } A_1 = \frac{2}{k};\ k > 1.$$

Using this, we conclude that

$$A_n = \frac{1}{k^n}\left(1 + \sum_{i=0}^{n-1} k^i\right) = \frac{k^n + k - 2}{k^n(k - 1)}$$

Thus,

$$A_2 = \frac{k + 2}{k^2};\quad A_3 = \frac{k^2 + k + 2}{k^3}, \text{ etc.}$$

$$\lim_{n\to\infty} A_n = \frac{1}{k - 1}.$$

Also solved by Paul S. Bruckman and the proposers.

Multiples of a Prime p

B-651 *Proposed by L. Van Hamme, Vrije Universiteit, Brussels, Belgium*

Let $u_0, u_1, \ldots$ be defined by $u_0 = 0$, $u_1 = 1$, and $u_{n+2} = u_{n+1} - u_n$. Also let p be a prime greater than 3, and for n in $X = \{1, 2, \ldots, p - 1\}$, let n^{-1} denote the v in X with $nv \equiv 1 \pmod{p}$. Prove that

$$\sum_{n=1}^{p-1} (n^{-1}u_{n+k}) \equiv 0 \pmod{p}$$

for all nonnegative integers k.

Solution by the proposer.

Let ρ be a zero of $1 + X + X^2$. Hence, $\rho^3 = 1$. Since

$$\begin{aligned}(1 + \rho)^p - 1 - \rho^p &= -\rho^{2p} - 1 - \rho^p \\ &= -(\rho^2 + 1 + \rho) = 0 \quad \text{if } p \equiv 1 \pmod{3} \\ &= -(\rho^{-2} + 1 + \rho^{-1}) = 0 \quad \text{if } p \equiv -1 \pmod{3},\end{aligned}$$

ρ is also a zero of $(1 + X)^p - 1 - X$. Hence,

$$\sum_{n=1}^{p-1}\binom{p}{n}\rho^n = 0, \quad \sum_{n=1}^{p-1}\binom{p}{n}\rho^{-n} = 0.$$

Multiplying the first equation with ρ^k, the second with ρ^{-k}, and using the easily verified formula

$$u_n = \frac{(-1)^{n-1}}{\sqrt{-3}}(\rho^n - \rho^{-n}),$$

we get

$$\sum_{n=1}^{p-1}(-1)^{n-1}u_{n+k}\binom{p}{n} = 0.$$

Dividing by p and using

$$\frac{1}{p}\binom{p}{n} \equiv \frac{(-1)^{n-1}}{n} \pmod{p}, \quad 1 \le n \le p-1,$$

we get the assertion.

Also solved by Paul S. Bruckman.

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(continued from page 288)

$Z_i(t)$ represents the number of zeros of f_t which are ε-close to η_i. By invariance of the complex integral, the functions $Z_i(t)$ are constant since the functions f_t vary continuously and do not vanish on the path of integration. Hence, $Z_i(0) = Z_i(1)$ for each i. This says that in a small neighborhood of each zero of f_1, there is a one-to-one correspondence of zeros of f_1 with zeros of f_0, in the required manner. □

In the case of our given functions, we find that the zeros of the polynomial $f_n(z)$ are close to the zeros of $g_n(z)$, which lie on the circle $|z| = \alpha$, as required, and the zeros of f_n get closer to the circle as $n \to \infty$. ■

Also solved by P. Bruckman, O. Brugia & P. Filipponi, L. Kuipers, and the proposer.

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