ON CERTAIN NUMBER-THEORETIC INEQUALITIES

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1. Introduction

This note deals with certain inequalities involving the elementary arithmetic functions $d(r)$, $\phi(r)$, and $\sigma_k(r)$ and their unitary analogues. We recall that a divisor d of r is called unitary [2] if $(d, r/d) = 1$. Let $e \equiv 1$, $I_k(r) = r^k$ $(k \geq 0)$, and μ denote the Möbius function. In terms of Dirichlet convolution, denoted by $(\cdot)$, we have [1]:

$$\left.\begin{aligned} d(r) &= (e \cdot e)(r) \\ \phi(r) &= (I \cdot \mu)(r) \\ \sigma_k(r) &= (I_k \cdot e)(r) \end{aligned}\right\} \text{ where } I(r) = r.$$

The unitary convolution of two arithmetic functions f and g is defined by

$$(1.1) \qquad (f \oplus g)(r) = \sum_{d \| r} f(d) g\left(\frac{r}{d}\right),$$

where $d \| r$ means that d runs through the unitary divisors of r. The unitary analogue μ^* of μ is given by [2]

$$(1.2) \qquad \mu^*(r) = (-1)^{\omega(r)},$$

where $\omega(r)$ denotes the number of distinct prime factors of r with $\omega(1) = 0$. The unitary analogue ϕ^* [2] of the Euler totient is given by

$$(1.3) \qquad \phi^*(r) = (I \oplus \mu^*)(r).$$

The unitary analogues of d and σ_k are d^* and σ_k^* and

$$(1.4) \qquad d^*(r) = 2^{\omega(r)},$$

$\omega(r)$ being as defined in (1.2);

$$\sigma_k^*(r) = (I_k \oplus e)(r).$$

For properties of σ_k^*, see [5]. It is known that d^*, ϕ^*, and σ_k^* are multiplicative functions. Further, given a prime p, $m \geq 1$,

$$(1.5) \qquad \begin{cases} d^*(p^m) = 2 \\ \phi^*(p^m) = p^m - 1 \\ \sigma_k^*(p^m) = p^{mk} + 1 \end{cases}$$

Let $\phi_k = (I_k \cdot \mu)$. $\phi_k(r)$ is multiplicative in r.

From the structure of ϕ_k and σ_k, we note that

$$\begin{aligned} (\phi_k \cdot \sigma_k) &= (I_k \cdot \mu) \cdot (I_k \cdot e) \\ &= (I_k \cdot I_k) \cdot (\mu \cdot e) \\ &= (I_k \cdot I_k) \text{ as } \mu \text{ is the Dirichlet inverse of } e. \end{aligned}$$

or

$$(1.6) \qquad \sum_{d | r} \phi_k(d) \sigma_k\left(\frac{r}{d}\right) = r^k d(r) \quad (k \geq 1).$$

It follows that

$$\phi_k(r) + \sigma_k(r) = \sum_{\substack{d|r \\ d \neq 1,\ d \neq r}} \phi_k(d)\sigma_k\left(\frac{r}{d}\right) = r^k d(r).$$

Therefore,

(1.7) $\phi_k(r) + \sigma_k(r) \leq r^k d(r)$

with equality if and only if r is a prime.

In arriving at (1.7), we have used the fact that ϕ_k and σ_k assume only positive values.

Defining $\phi_k^* = I_k \oplus \mu^*$, and noting that

(1.8) $\phi_k^* \oplus \sigma_k^* = r^k d^*(r)$,

we have

Theorem 1: $\phi_k^*(r) + \sigma_k^*(r) \leq r^k d^*(r)$ with equality if and only if r is a prime power.

Further, using the fact that

$$\phi_k^* \oplus d^* = \sigma_k^*,$$

we also obtain

Theorem 2: $\phi_k^*(r) + d^*(r) \leq \sigma_k^*(r)$ with equality if and only if r is a prime power.

We remark that Theorem 2 is analogous to the inequality involving ϕ, d, and σ, see [4], [6].

Using the multiplicativity of ϕ_k^* and σ_k^*, one could also prove

Theorem 3: For $k \geq 1$,

$$\frac{1}{\zeta(2k)} < \frac{\sigma_k^*(r)\phi_k^*(r)}{r^2} < 1,$$

where $\zeta(s)$ is the Riemann ζ-function.

Now, the AM-GM inequality yields

(1.9) $\dfrac{\sigma_k(r)}{d(r)} \geq r^{k/2}$ (see [9])

and

(1.10) $\dfrac{\sigma_k^*(r)}{d^*(r)} \geq r^{k/2}$.

The aim of this note is to establish a few more inequalities which come out as special cases of certain general inequalities found in [3] and [7].

Let

$$0 < a \leq a_i \leq A \qquad (i = 1, 2, \ldots, s)$$
$$0 < b \leq b_i \leq B$$

where a_i, b_i ($i = 1, 2, \ldots, s$), a, A, b, B are real numbers. Then, from [7],

(1.11) $$\frac{\left(\sum_{i=1}^{s} a_i^2\right)\left(\sum_{i=1}^{s} b_i^2\right)}{\left(\sum_{i=1}^{s} a_i b_i\right)^2} \leq \frac{(AB + ab)^2}{4ABab}$$

Next, let

$$0 \leq a_1^{(k)} \leq a_2^{(k)} \leq \cdots \leq a_s^{(k)} \quad (k = 1, 2, \ldots, m).$$

Then, an inequality due to Tchebychef [3] states that:

$$(1.12) \quad \left(\frac{\sum_{i=1}^{s} a_i^{(1)}}{s}\right) \cdots \left(\frac{\sum_{i=1}^{s} a_i^{(m)}}{s}\right) \le \frac{\sum_{i=1}^{s} a_i^{(1)} \cdots a_i^{(m)}}{s}$$

The inequalities derived in Section 2 are essentially illustrations of (1.11) and (1.12).

2. Inequalities

Theorem 4: For $k \ge 0$,

$$(2.1) \quad \frac{\sigma_k(r)}{d(r)} \le \frac{r^k + 1}{2}$$

and

$$(2.2) \quad \frac{\sigma_k^*(r)}{d^*(r)} \le \frac{r^k + 1}{2}.$$

Proof of (2.1): Let $d_1, \ldots, d_s$ be the divisors of r. We appeal to (1.11) by taking $a_i = d_i^{k/2}$, $b_i = d_i^{-m/2}$, $A = r^{k/2}$, $a = 1$, $b = r^{-m/2}$, $B = 1$. Then

$$\frac{\sigma_k(r)\sigma_m(r)}{r^m\left(\sigma_{(k-m)/2}^{(r)}\right)^2} \le \frac{(r^{k/2} + r^{-m/2})^2}{4r^{k/2 - m/2}}$$

or

$$(2.3) \quad \frac{(\sigma_k(r)\sigma_m(r))^{1/2}}{\sigma_{(k-m)/2}^{(r)}} \le \frac{1}{2r^{(k-m)/2}}(r^{(k+m)/2} + 1)$$

Setting $m = k$ in (2.3), we obtain (2.1).

Similarly, by considering the unitary divisors of r, we arrive at (2.2). In view of (1.9) and (1.10), we also have

Corollary:

$$(2.4) \quad r^{k/2} \le \frac{\sigma_k(r)}{d(r)} \le \frac{r^k + 1}{2}$$

and

$$(2.5) \quad r^{k/2} \le \frac{\sigma_k^*(r)}{d^*(r)} \le \frac{r^k + 1}{2}$$

Theorem 5: For $k, m \ge 0$,

$$(2.6) \quad \frac{\sigma_{k+m}(r)}{\sigma_m(r)} \ge r^{k/2}$$

and

$$(2.7) \quad \frac{\sigma_{k+m}^*(r)}{\sigma_m^*(r)} \ge r^{k/2}.$$

Proof of (2.6): Let $d_1, \ldots, d_s$ be the divisors of r. We appeal to (1.12) with

$$a_i^{(1)} = d_i^{k_1}, \ldots, a_i^{(m)} = d_i^{k_m} \quad (i = 1, 2, \ldots, s)$$

where $k_1, \ldots, k_m$ are positive numbers.

Then,

$$\frac{\sigma_{k_1 + \cdots + k_m}(r)}{s} \ge \frac{\sigma_{k_1}(r)}{s} \cdots \frac{\sigma_{k_m}(r)}{s}$$

with $s = d(r)$. From (1.9), we obtain

$$(2.8) \quad \frac{\sigma_{k_1 + \cdots + k_m}(r)}{\sigma_{k_i}(r)} \ge r^{\frac{1}{2}\sum_{j \ne i} k_j}.$$

Writing $m = 2$, we get

$$\frac{\sigma_{k_1+k_2}(r)}{\sigma_{k_2}(r)} \geq r^{k_1/2},$$

which proves (2.6).

The proof of (2.7) is similar and is omitted here.

Remark: Inequalities (2.6) and (2.7) generalize (1.9) and (1.10), respectively.

In this connection, we point out that analogous to the inequality $\phi(r)d(r) \geq r$ [8], one could prove using multiplicativity of ϕ_k^* and d^* that

Theorem 6: For $k \geq 1$,

(2.9) $\quad d^*(r)r^k \leq \phi_k^*(r)(d^*(r))^2 \leq r^{2k}.$

The proof of (2.9) is omitted.

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