GENERALIZATIONS OF THE DUAL ZECKENDORF INTEGER REPRESENTATION THEOREMS—DISCOVERY BY FIBONACCI TREES AND WORD PATTERNS

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"What is the use," thought Alice, *"of a book without pictures and conversations?..."*

Alice's Adventures in Wonderland,
—Lewis Carroll

1. Introduction

In this paper we show how the two well-known integer representation theorems which are associated with the name of Zeckendorf may be generalized as dual systems by constructing colored tree sequences whose shade sets partition $Z^+ = \{1, 2, \ldots\}$. Many interesting properties of the representations can be observed directly from the tree diagrams, and the proofs of the properties can truly be said to be "evident" or "obvious"; we shall not translate such proofs into other symbolic forms.

The Zeckendorf theorems are about representations of positive integers as sums of distinct elements of given number sequences. The first theorem is in Lekkerkerker [6], and a dual of it is given by Brown [2]. Early papers on properties of the Zeckendorf integer representations are Zeckendorf [12] and Brown [1]. Klarner [5] gives an excellent review of the literature to 1966, and extends many of the theories to that date. In [3] Carlitz et al. (1972) define Fibonacci representations of integers, and study their properties.

In Turner [7] we showed how to construct certain tree sequences and defined their *shade sets*, which together demonstrated the Zeckendorf representation theorems. In Zulauf & Turner [13], we showed how the shade sets could be defined in a set-theoretic notation, and proved the Zeckendorf theorems in a concise manner. In Turner [8] and Turner & Shannon [9] we defined Fibonacci word patterns and used them to study tree shade sets.

Notation and definitions for integer representations

(i) Let $\mathbf{c} = \{c_1, c_2, c_3, \ldots\}$ be any sequence of distinct real numbers, and let $N \in Z^+$ (i.e., N is a positive integer). We shall be concerned with representations of N of the form

$$N = \sum_{i=1}^{n} e_i c_i, \tag{1.1}$$

where $n \geq 1$ and $e_i \in \{0, 1\}$ for each i.

In this paper $\mathbf{c}$ will be a strictly increasing sequence of nonnegative integers. Once $\mathbf{c}$ is given, the vector $\mathbf{e} = \{e_1, \ldots, e_n\}$ determines the representation.

(ii) (as in [4]): The sequence $\mathbf{c}$ is *complete* with respect to the positive integers if and only if every integer $N \in Z^+$ has a representation of the form (1.1).

(iii) (see [4]): If the number of elements of $\mathbf{c}$ used in a representation is as small as possible, the representation is said to be *minimal*; if it is the largest possible, then the representation is *maximal*.

The dual Zeckendorf theorems

We shall use the notation Z and Z^* when referring to these theorems and related properties.

Theorem 1.1 (Z; [6], [12]): Every $N \in Z^+$ has one and only one representation in the form (1.1) with $\mathbf{c} = \{u_n\} = \{F_{n+1}\} = \{1, 2, 3, 5, \ldots\}$, the Fibonacci sequence, and with the coefficients e_i satisfying $e_n = 1$, and $e_i e_{i+1} = 0$ if $1 \le i < n$.

Moreover, these representations are minimal; and for a given value of n there are F_n integers having Zeckendorf representations.

Theorem 1.2 (the dual, Z^*; [2]): Every $N \in Z^+$ has one and only one representation in the form (1.1) with $\mathbf{c} = \{u_n\}$, $e_i \in \{0, 1\}$, $e_n = 1$, and $e_i + e_{i+1} \ne 0$ if $1 \le i < n$.

Moreover, these representations are maximal; and for a given value of n there are F_{n+1} integers having Zeckendorf representations.

The generalization in this paper:

In Section 2 we show how to construct classes of colored tree sequences whose shade sets exactly cover Z^+, and hence derive classes of complete sequences of integers (those used to color the trees). From these classes we select two which are dual in a sense that generalizes the dual conditions $e_i e_{i+1} = 0$ and $e_i + e_{i+1} \ne 0$ used in the Zeckendorf theorems 1.1 and 1.2, respectively. Thus, we obtain a class of dual integer representation theorems, of which the pair (Z, Z^*) is the simplest case.

2. Colored Trees and Their Shades

Definitions

(i) A *tree* is a set of n nodes (or points), and a set of $n - 1$ *edges* (lines joining pairs of the nodes), having no *cycles* (paths from a node which return to that node).

(ii) If one node in a tree is distinguished, and labelled as a *root*, we have a *rooted tree*.

(iii) If real numbers (in this paper integers) are assigned to the nodes of a rooted tree, we have a *number tree*. We call the numbers *colors* of the nodes.

(iv) A node, other than the root, which has only one edge attached to it is called a *leaf*. There is a unique path from the root to any given leaf. The sum of the colors on a root-to-leaf path is called the *shade* of the path.

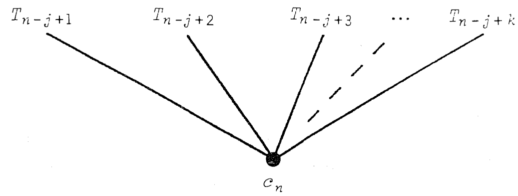
(v) The set of shades of all root-to-leaf paths in a rooted tree is the *shade set* (or *shade*) of the tree.

Generation of a 3-parameter class of colored tree sequences

Suppose we are given a coloring sequence of integers, denoted by $\mathbf{c} \equiv c_0, c_1, c_2, \ldots, c_n, \ldots$; and also an initial sequence of i rooted trees denoted by $T_0, T_1, \ldots, T_{i-1}$, each of whose nodes is colored by a member of $\mathbf{c}$.

Then we can continue the tree sequence in the following way.

For the n^{th} tree, take a k-fork (with $k \le n$) and color its root node c_n. Select an ordered subsequence of the $T_0, T_1, \ldots, T_{n-1}$, of length k and using consecutive members, and mount them one by one from left to right on the k prongs on the fork. The following diagram makes this construction clear:

(2.1) $T_n \equiv$ 

with $k < j \leq n + 1$.

Any selection of values for the triple (i, j, k) will determine a sequence of colored trees, so the construction just defined determines a 3-parameter family of such trees. We shall also allow j, k to be functions of n.

Tree sequences with shade sets exactly equal to Z_0^+:

We investigate now the choices of **c**, the triple (i, j, k), and the initial trees such that they will lead to a sequence of trees having shade set $Z_0^+ = \{0, 1, 2, \ldots\}$. We shall require this to happen *exactly*; which is to say that if Z_m denotes the shade set of the m^{th} tree T_m of a sequence we shall require

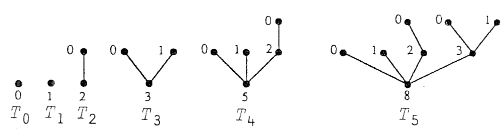
$$\bigcup_{m=0}^{\infty} Z_m = Z_0^+ \quad \text{and} \quad Z_u \cap Z_v = \emptyset \text{ when } u \neq v.$$

Examples

Before giving a general result, we shall give three examples to illustrate the various concepts introduced above. The first two provide graphical proofs of the dual Zeckendorf theorems; we treated these in [7] and [13]. The third gives an indication of the generalization we are aiming at, and we give the first seven trees of its sequence.

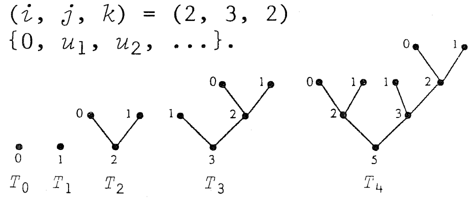
Example 1 (Zeckendorf, Z)

Parameter values: $(i, j, k) = (2, n + 1, n - 1)$ (for T_n)
Color sequence: $\{0, u_1, u_2, \ldots\}$ where $u_n = F_{n+1}$.



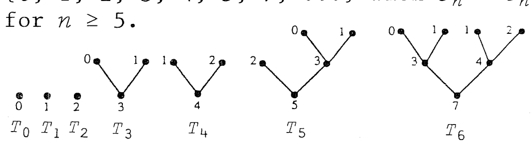
Example 2 (Zeckendorf dual, Z^*)

Parameter values: $(i, j, k) = (2, 3, 2)$
Color sequence: $\{0, u_1, u_2, \ldots\}$.



Example 3 (gap range 1, 2)

Parameter values: $(i, j, k) = (3, 4, 1)$
Color sequence: $\{0, 1, 2, 3, 4, 5, 7, \ldots\}$ with $c_n = c_{n-2} + c_{n-3}$, for $n \geq 5$.



For this last example, it may be observed that the sequence of tree shade sets $\{Z_n\}$, $n = 0, 1, 2, \ldots$ is

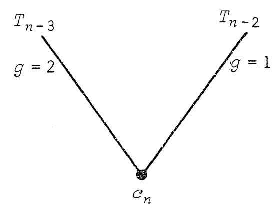
$$\{\{0\}, \{1\}, \{2\}, \{3, 4\}, \{5, 6\}, \{7, 8, 9\}, \{10, 11, 12, 13\}, \ldots\}.$$

It is also seen that the number of leaves in T_n is c_{n-2} after $n = 5$. It should be evident that with this color sequence and method of tree construction the shade set sequence will continue through the positive integers. Thus, the shade set covers Z_0^+ exactly. So, given any positive integer N, it will correspond to the shade of just one root-to-leaf path in one tree of this sequence of colored rooted trees. It is easy to derive a formula to tell which. We shall not do this here, but rather remark upon the fact that the numbers (colors) on that root-to-leaf path constitute a representation for N as a sum of distinct members of $\mathbf{c}$. Thus, $\mathbf{c}$ is complete for Z^+. Moreover, because of the parameter values (i, j, k) and the construction process, we can state, from the following illustration, that the representation for N has gaps in its constituent colors of either 1 or 2 (i.e., at least a gap of 1 and at most a gap of 2).

Take $N = 12$, for example. This occurs in the shade of T_7. The third root-to-leaf path from the left gives the representation

$$12 = 1 + 4 + 7 = c_1 + c_4 + c_6.$$

The binary representation (i.e., the vector of e-values) is (0, 1, 0, 0, 1, 0, 1). The "gaps" referred to above can now be seen as runs of 0's occurring between the 1's. All representations from this tree sequence will have a gap of 1 zero or 2 zeros between every pair of adjacent 1's. A graphical "proof" of this is to write the construction rule thus, where the color gap sizes occurring are indicated on the fork edges:

$T_n =$ 

, for $n \geq 3$

It is evident that in every tree beyond the third a 2 or a 1 must occur on every edge, and hence only gaps of 2 and 1 occur in all root-to-leaf path sums.

The reader may care to check that similar reasoning applied to the trees of Examples 1 and 2 will verify the dual Zeckendorf theorems, with their gap properties that $e_i e_{i+1} = 0$ and $e_i + e_{i+1} \neq 0$, respectively.

Before going on to define the dual classes which generalize the Zeckendorf theorems, we give an indication of how studies of Fibonacci word patterns [8] occurring on the tree sequences can provide theorems about properties of integer representations. Referring to Example 2, for instance, suppose we wish to investigate the occurrence of integer representations with the Zeckendorf dual properties and which contain $u_1 = 1$ (i.e., also $e_1 = 1$). Examining the trees, we see that $u_1 = 1$ occurs only on leaf nodes. It is easy to derive formulas for the number of u_1's occurring in tree T_n (it is obviously F_n), and for the pattern of the occurrences. Details of the pattern are given in [8]; briefly, the pattern starting with T_1 is given by the Fibonacci word juxtaposition recurrence formula

$$W_{n+2} = W_n W_{n+1}, \text{ with } W_1 = 1 \text{ and } W_2 = 01.$$

This gives the pattern (which is the leaf-node color pattern) 1, 01, 101, 01101, ... The positions of the 1's are the places 1, 3, 4, 6, 8, 9, ... and of the 0's are 2, 5, 7, 10, ... These two sequences are the well-known Wythoff number sequences, given by $\{[\alpha n]\}$ and $\{[\alpha^2 n]\}$, respectively, where $\alpha = \frac{1}{2}(1+\sqrt{5})$ and $[x]$ is the greatest integer function ([8] and [11]). Similar analyses lead to similar conclusions for the placings of the other colors in the tree sequence.

To study relative positions and frequencies of occurrences of integer representations from Example 3, it is necessary to solve the third-order recurrence given for c_n; and to study the corresponding word pattern recurrence $W_{n+3} = W_n W_{n+1}$ with initial words $W_1 = 0$, $W_2 = 1$, $W_3 = 2$.

3. Generalized Dual Zeckendorf Theorems

We choose parameter values for (i, j, k) in (2.1) so that two dual tree-sequence classes are defined. With suitable choices of initial trees, and of coloring sequences, we shall ensure that the first one (designated the GZ-class) generates integer representations such that all gaps in the e-vectors have *at least* g^* *zeroes*; and that the second one (designated the GZ^*-class) generates integer representations with all gaps having *at most* g^* *zeroes*. To be precise, we define a *gap* g to be a run of g zeroes occurring between two successive 1's in an e-vector. The conditions "at least g^*" and "at most g^*" on the gap sizes in the e-vector representations are the dual conditions. We note immediately that the GZ-class will contain the sequence of Example 1, since the conditions $e_j e_{j+1} = 0$ and $g \geq 1$ are equivalent. Likewise, the GZ^*-class contains the sequence of Example 2, since the conditions $e_j + e_{j+1} \neq 0$ and $g \leq 1$ are equivalent.

The following tables give definitions, and the first few color sequences and corresponding tree sequences as examples.

TABLE 1. Definitions

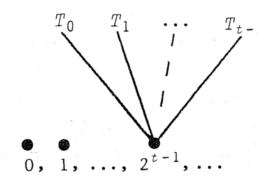
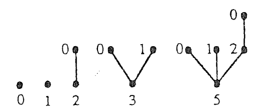
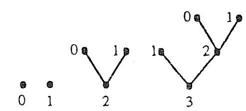
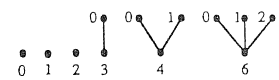
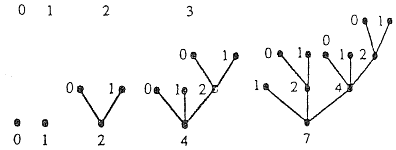
GZ-class	GZ^*-class
Gap sizes: $g \geq g^*$	$g \leq g^*$
Parameter: $(i, j, k) = (g^* + 1, n + 1, n - g^*)$ (for T_n)	$(i, j, k) = (g^* + 1, g^* + 2, g^* + 1)$
Color sequence: $c_{n+i} = c_n + c_{n+i-1}$	$c_{n+i} = c_n + c_{n+1} + \cdots + c_{n+i-1}$
Initial colors: $0, 1, 2, \ldots, i$	$0, 2^0, 2^1, 2^2, \ldots, 2^{i-1}$
Initial trees: $\{\underset{r}{\bullet} \mid r = 0, \ldots, i-1\}$	The first $i+1$ trees of $\{T_n\}$ are given by:  with $2 \leq t \leq i$.
General solution for **c** is given in §4	General solution for **c** is given in §4

TABLE 2. Example Sequences

g^*	GZ-Class	GZ^*-Class
1	0, 1, 2; 3, 5, 8, 13, ...	0, 1, 2; 3, 5, 8, 13, ...
2	0, 1, 2, 3; 4, 6, 9, 13, ...	0, 1, 2, 4; 7, 13, 24, 44, ...
3	0, 1, 2, 3, 4; 5, 7, 10, 14, ...	0, 1, 2, 4, 8; 15, 29, 56, ...
	Tree Sequences ($g \geq g^*$)	Tree Sequences ($g \leq g^*$)
1		
2		

It should be noted that, since the shades of all the tree sequences in both the GZ- and GZ^*-classes exactly cover Z_0^+, all the color sequences used (with $g_0 = 0$ deleted) are complete for Z^+.

Within each pair of tree sequences, for each value of g^*, the root-to-leaf paths give integer representations using distinct colors and with gaps satisfying $g \geq g^*$ and $g \leq g^*$, respectively. Those on the tree sequences with $g^* = 1$ are the dual Zeckendorf representations.

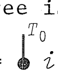
As we said in the Introduction, there is hardly a need for formal proofs of the above statements about the integer representation properties. They all follow by induction, using the definitions of the procedures for constructing the colored trees. Study of the general tree diagram tells all! As Alice thought, in Wonderland: "What is the use of a book without pictures... ." However, to demonstrate the reason for the choices of (i, j, k) in the two classes, we shall give some details of the proofs. The key property to establish is that the shade sets of the trees in any sequence partition Z_0^+.

Theorem: Each tree-sequence in the two classes defined in Table 1 has shade set exactly equal to $Z_0^+ = \{0, 1, 2, \ldots\}$.

Proof: We shall use induction, for sequences in each class.

Case (i) Let $T = \{T_n\}$ be a tree-sequence in the GZ-class.

The first i trees in T have shades $0, 1, \ldots, (i - 1)$, respectively, by the definitions of initial colors and initial trees given in Table 1. The $(i + 1)^{\text{th}}$ tree is

$T_i =$  i,

since $k = n - g = i - (i - 1) = 1$ (meaning there is a 1-fork), and $n - j + 1 = n - (n + 1) + 1 = 0$ (meaning that T_0 is mounted on it). Here we have used the formulas given in Table 1 for the parameters (i, j, k) in the GZ-class.

Thus, T_i has shade $0 + i = i$, which continues the shade sequence required by the theorem.

We now make the inductive hypothesis that the shade sets continue as for the theorem, up to the last (rightmost) branch of tree T_n, with $n > i$.

Referring to the construction diagram (2.1), inserting parameters $j = n + 1$ and $k = n - g^*$, we find that T_0 is mounted on the first (leftmost) branch of the k-fork used to construct T_n. This is also true for T_{n+1}, etc. Hence, the leftmost branch shade of T_{n+1} is $c_{n+1} + 0 = c_{n+1}$.

Now, the rightmost branch shade of tree T_n is c_n + (rightmost branch shade of T_{n-g^*-1}), which, by the inductive hypothesis, equals $c_n + (c_{n-g^*} - 1)$. Then, since $c_n + c_{n-g^*} = c_n + c_{n-i+1} = c_{n+1}$ (using parameter and color sequence definitions), we have shown that

$$(\text{leftmost branch shade of } T_{n+1}) = (\text{rightmost branch shade of } T_n) + 1.$$

Hence, the shade of T_{n+1} follows on in natural sequence from that of T_n. This completes the inductive proof.

Case (ii) Let $T = \{T_n\}$ belong to the GZ^*-class.

We proceed as for Case (i); we shall omit the details showing that the shades of $T_0, T_1, \ldots, T_{i+1}$ conform to the theorem.

Assume that the shade of the tree sequence $T_0, T_1, \ldots, T_n$, with $n > i + 1$, is a sequence $0, 1, 2, \ldots, r$. We shall show that the first element of the shade of T_{n+1} is $r + 1$.

Let us use the notation L_n, R_n to mean, respectively, the "leftmost branch shade of tree T_n" and the "rightmost branch shade of tree T_n." We have to show that $R_n + 1 = L_{n+1}$. From the construction diagram (2.1), and inserting the parameters for j, k from Table 1 for the GZ^*-class, we see that

$$R_n = c_n + R_n \quad (= c_n + L_n - 1);$$

and

$$L_{n+1} = c_{n+1} + L_{n-g^*} \ (= c_n + c_{n-1} + \cdots + c_{n-g^*} + L_{n-g^*}).$$

Now

$$L_n = c_n + L_{n-g^*-1} = c_n + (L_{n-g^*} - c_{n-g^*-1}),$$

using the fact that, for $n \geq 1$, the cardinal number of the shade set of T_n is equal to c_n: this is easily established by induction, for the trees in GZ^*. So we have

$$\begin{aligned} R_n + 1 = c_n + L_n &= 2c_n - c_{n-g^*-1} + L_{n-g^*} \\ &= c_n + c_{n-1} + \cdots + c_{n-g^*} + L_{n-g^*} \\ &= c_{n+1} + L_{n-g^*} \\ &= L_{n+1}. \end{aligned}$$

The existence of the generalized dual Zeckendorf integer representations now follows immediately. The proof that the gap sizes satisfy conditions $g \geq g^*$ or $g \leq g^*$ for tree sequences in the GZ-class or GZ^*-class, respectively, rests on simple observations of the gaps that can occur [see diagram (2.1)] between c_n and the root colors of the k trees $T_{n-j+1}, \ldots, T_{n-j+k}$ used in the construction.

The final table gives the dual representations of $N = 1, 2, \ldots, 10$ for the cases $g^* = 1$ and $g^* = 2$. Note that they are, respectively, minimal and maximal representations. (See Table 3 below.)

4. Formulas for the Color Sequences in the Two Classes

In Table 1 we gave the initial values and general recurrence equations for the color sequences in the GZ- and GZ^*-classes. We end the paper by giving general solutions for the equations, which provide formulas for the terms of the sequences in terms of weighted sums of binomial coefficients. We also give geometrical interpretations for these weighted sums: they are related to the elements on certain diagonals of Pascal's triangle. Thus, in a very nice pictorial way, we have linked the generalizations of the dual Zeckendorf integer representations to generalizations of the Pascal-Lucas theorem which states that sums of the terms on the 45° upward diagonals of Pascal's triangle are Fibonacci numbers.

TABLE 3. Dual Integer Representations

	GZ-Class			*GZ**-Class		
N	$g^* = 1$	$g^* = 2$	N	$g^* = 1$	$g^* = 2$
1	1 (1)	1 (1)	1	1 (1)	1 (1)
2	2 (01)	2 (01)	2	2 (01)	2 (01)
3	3 (001)	3 (001)	3	1 + 2 (11)	1 + 2 (11)
4	1 + 3 (101)	4 (0001)	4	1 + 3 (101)	4 (001)
5	5 (0001)	1 + 4 (1001)	5	2 + 3 (011)	1 + 4 (101)
6	1 + 5 (1001)	6 (00001)	6	1 + 2 + 3 (111)	2 + 4 (011)
7	2 + 5 (0101)	1 + 6 (10001)	7	2 + 5 (0101)	1 + 2 + 4 (111)
8	8 (00001)	2 + 6 (01001)	8	1 + 2 + 5 (1101)	1 + 7 (1001)
9	1 + 8 (10001)	9 (000001)	9	1 + 3 + 5 (1011)	2 + 7 (0101)
10	2 + 8 (01001)	1 + 9 (100001)	10	2 + 3 + 5 (0111)	1 + 2 + 7 (1101)

The recurrence for the *GZ*-class

We wish to index and refer to the sequences in the *GZ*-class, as g^* ranges over 1, 2, 3, etc. To this end we add a superscript in brackets, to the expression for the n^{th} term in the $g^{*\text{th}}$ sequence. Thus, $c_n^{(g^*)}$ denotes this expression. Since it is typographically clumsy to use g^* as the indexing letter, we shall replace g^* by i (note that the values for i as used here and subsequently are 1 less than the ones used for the parameter in Table 1).

The recurrence equation for $c_n^{(i)}$ (omitting $c_0^{(i)} = 0$ from each sequence) is

(4.1) $$c_{n+i+1}^{(i)} = c_{n+i}^{(i)} + c_n^{(i)}, \text{ for } n \geq 1,$$

with initial values

(4.2) $$c_s^{(i)} = s \text{ for } s = 1, 2, \ldots, i + 1.$$

[Note that for $i = 1$ this gives the Fibonacci sequence = $\{F_{n+1}\}$.]

The *general solution* to (4.1) and (4.2), given i, is

(4.3) $$c_{n-i+1}^{(i)} = \sum_{r=0}^{[n/i]} \binom{n - ri}{r}, \text{ for } n - i + 1 = 1, 2, 3, \ldots,$$

which can be demonstrated by direct substitution, and making use of the identity

$$\binom{n+1}{r+1} = \binom{n}{r+1} + \binom{n}{r}$$

for the binomial coefficients.

[We use the normal convention that $\binom{a}{b} \equiv 0$ when $a < b$.]

If i is small, say $i = 1, 2, 3,$ or 4, then we can use Binet-type formulas to calculate the $c_n^{(i)}$ efficiently for any n. If i is large, then formula (4.3) above is probably the most efficient way to calculate $c_n^{(i)}$ exactly.

For example, if $i = 1000$, then $c_{2200}^{(1000)}$ is

$$\binom{3199}{0} + \binom{2199}{1} + \binom{1199}{2} + \binom{199}{3},$$

which equals

$$1 + 2199 + 1199 \times 1198/2 + 199 \times 198 \times 197/6 = 2014100.$$

Thus, to calculate $c_{2200}^{(1000)} = 2014100$ from the above formula requires only a few additions and multiplications; whereas to calculate it directly from the recurrence relation (4.1) could require that all the $c_n^{(1000)}$ be precomputed for $n = 1, 2, \ldots, 2199$. Clearly, formula (4.3) is much quicker.

The solution (4.3) also has a nice geometric interpretation, which we show in the final subsection.

The recurrence for the GZ^*-class

The recurrence equation for $c_n^{*(i)}$ (omitting $c_0^{*(i)} = 0$ from each sequence) is

(4.4) $\quad c_{n+i+1}^{*(i)} = c_n^{*(i)} + c_{n+1}^{*(i)} + \cdots + c_{n+1}^{*(i)}$, for $n \geq 1$,

with initial values

(4.5) $\quad c_s^{*(i)} = 2^{s-1}$ for $s = 1, 2, \ldots, i + 1$.

[Note that $i = 1$ again gives the Fibonacci sequence $\{F_{n+1}\}$.]

By considering $c_{n+i+1}^{*(i)} - c_{n+i}^{*(i)}$, and using (4.4), we find that

(4.6) $\quad c_{n+i+1}^{*(i)} = 2c_{n+i}^{*(i)} - c_{i-1}^{*(i)}$, $n = 1, 2, \ldots$.

We used this equivalent form of the recurrence equation as a first step in obtaining a general solution. The details of our solution method are lengthy, and will be reported elsewhere. Our solution is given next, as (4.7): it may be checked by insertion into (4.4) or (4.6) and use of elementary algebra and manipulations with binomial coefficients.

The *general solution* to (4.4) and (4.5), given i, is

$$c_n^{*(i)} = u_n^{(i+1)} - u_{n-1}^{(i+1)} \text{ for } n = 1, 2, \ldots,$$

where

(4.7) $$u_n^{(i)} = \sum_{r=0}^{[n/i]} \frac{2^{n-ri}}{(-2)^r}\binom{n - ri}{r} \text{ for } n = 0, 1, 2, \ldots .$$

We show in the next subsection how the solutions for the dual pairs of recurrences in the GZ- and GZ^*-classes are neatly related to elements on the diagonals of increasing slope within Pascal's triangle.

Geometric interpretation

Consider Pascal's triangle, for the binomial coefficients, drawn as a 45° triangle rather than the usual equilateral triangle, thus:

$c_7^{(3)}$

$13 = c_6^{(1)}$, $21 = c_7^{(1)}$ } Lucas diagonal sums

n										
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	2	7			
8	1	8	28	56	70	56	28	8	1	
9	1	9	36	84	126	126	84	36	9	1

Then $c_{n+1-i}^{(i)}$ is just the sum of all the binomial coefficients on the line of *slope* i that starts on the left end of the n^{th} row. In particular, the n^{th} Fibonacci number is the sum of the numbers on a 45° line (slope $i = 1$) starting at the n^{th} row (these lines are the well-known Lucas diagonals). As an example, for the case $i = 3$,

$$c_7^{(3)} = c_{9+1-3}^{(3)} = \binom{9}{0} + \binom{6}{1} + \binom{3}{2} = 1 + 6 + 3 = 10.$$

The geometric interpretation also suggests the results $c_n^{(0)} = 2$ and $c_n^{(\infty)} = n$ corresponding to lines of slope 0 (horizontal) and ∞ (vertical), respectively.

As an example in the GZ*-class, again taking $i = 3$, and with $n = 9$, we get

$$c_9^{*(3)} = u_9^{(4)} - u_8^{(4)} = \left[2^9\binom{9}{0} - \frac{2^5}{2}\binom{5}{1} - 2^8\binom{8}{0} - \frac{2^4}{2}\binom{4}{1}\right]$$

$$= 432 - 224 = 208.$$

Inspection of Pascal's triangle shows that $u_n^{(i)}$ is a weighted sum of the elements on the upward diagonal of slope i which begins at the first element of the n^{th} row: the weights are powers of 2 as given in (4.7).

Hence, $c_n^{*(i)}$ is the difference of weighted sums from the adjacent diagonals beginning on the n^{th} and $(n - 1)^{\text{th}}$ rows.

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