CHARACTERISTICS AND THE THREE GAP THEOREM

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1. Introduction

In order to determine the sequence $v = ([m\alpha],\ m = 1, 2, 3, \ldots)$, for irrational α (where $[x]$ denotes the largest integer not exceeding x), Bernoulli [1] considered the sequence of differences $d_1, d_2, d_3, \ldots$, where

(1) $\quad d_m = [(m+1)\alpha] - [m\alpha],\ m = 1, 2, 3, \ldots .$

Clearly then,

$$[m\alpha] = \sum_{i=1}^{m-1} d_i + [\alpha],\ m = 3, 4, 5, \ldots .$$

Thus, knowing the first two terms of v, one can then determine the entire sequence from (1). For example, with $\alpha = \sqrt{2}$, we have the following.

m	d_m	$[m\alpha]$
1	1	1
2	2	2
3	1	4
4	2	5
5	1	7
6	1	8
7	2	9
8	1	11
9	2	12
10	1	14

It may be shown that d_m may only equal $[\alpha]$ or $[\alpha] + 1$ (that is, 0 or 1 when $0 < \alpha < 1$). If we replace $[\alpha]$ by s (small) and $[\alpha] + 1$ by l (large), then we obtain a string of such characters. This we will refer to as the *characteristic* of α. For example, the characteristic of $\alpha = \sqrt{2}$ is $slslsslsl\ldots$.

String operations may be used to generate the characteristic from its first few terms, by utilizing the continued fraction expansion of α. Bernoulli was the first to guess the rules which were the basis of these string operations. These were reformulated in a more attractive form by Christoffel [2]. However, it had to wait until Markoff [9] before the first proofs were offered. In Section 4 we show how the characteristic is generated.

In this paper we demonstrate a rather intriguing connection between the characteristic of α and the sequence of arcs or gaps formed by the partition of the circle by the successive placement of points by the angle α revolutions. The connection is not immediately obvious and does not hold for all values of α. We use results from the Three Gap Theorem, a result first conjectured by Steinhaus (see [6, 10, 11, 13-15, 18, 19]) which states that N points placed on the circle as above partition it into gaps of either three or two different lengths.

Consider such a circle when N is equal to the denominator of a convergent [see (2)] to α. Only in this case is the circle partitioned into gaps of exactly two different lengths. We can label these gaps as large or small, assigning l or s where appropriate and thus we have a string of gap types, ordered clockwise about the circle, with the first element describing the gap adjacent to the origin.

We show that when N is the denominator of a *total* convergent [see (3)] this string, after a trivial permutation, forms the first few terms of the characteristic, but *only* for special values of α (for example, those numbers with identical terms in their continued fraction expansion). One such value is the golden number, $\alpha = \tau = (\sqrt{5} - 1)/2$. The golden number's characteristic has interesting properties (see [16]) and we give it a special name—the *Golden Sequence*.

In order to state Christoffel's rule for generating the characteristic, we introduce in Section 2 some aspects from the theory of continued fractions. The Three Gap Theorem is later described in more detail in Section 3 before we prove our main result (in Section 4.2).

2. Continued Fractions

Write $t_0 = \alpha$ and express (for $n = 0, 1, 2, \ldots$),

$$a_n = [t_n],$$

$$t_{n+1} = \frac{1}{\{t_n\}},$$

where $\{x\} = x - [x]$ is the fractional part of x. Thus, we can generate the simple continued fraction expansion of α, namely,

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

$$= \{a_0;\, a_1,\, a_2,\, a_3,\, \ldots\}.$$

The *partial* convergents to α are defined as

(2) $$\frac{p_{n,i}}{q_{n,i}} = \{a_0;\, a_1,\, a_2,\, \ldots,\, a_{n-1},\, i\}, \quad i = 1, 2, \ldots, a_n - 1,$$

while

(3) $$\frac{p_{n,a_n}}{q_{n,a_n}} = \frac{p_n}{q_n} = \{a_0;\, a_1,\, a_2,\, \ldots,\, a_{n-1},\, a_n\},$$

defines the *total* convergents.

For example, the continued fraction of τ is given by

$$\tau = \{0;\, 1 + \tau\} = \{0;\, 1,\, 1 + \tau\} = \{0;\, 1,\, 1,\, 1,\, \ldots\}.$$

All convergents to τ are total convergents and

$$p_n = q_{n-1} = F_n = F_{n-1} + F_{n-2},\ n \geq 1,\ F_{-1} = 1,\ F_0 = 0.$$

We quote some results from the theory of continued fractions (see Khintchine [7]);

(4) $$\frac{p_{n,i}}{q_{n,i}} = \frac{p_{n-2} + ip_{n-1}}{q_{n-2} + iq_{n-1}},\ p_{-2} = q_{-1} = 0,\ q_{-2} = p_{-1} = 1,$$

(5) $$p_{n-1}q_{n,i} - q_{n-1}p_{n,i} = (-1)^n,$$

(6) $$q_n\alpha - p_n = \frac{(-1)^n}{t_{n+1}p_n + p_{n+1}},$$

(7) $$q_{n,i}\|q_{n-1}\alpha\| + q_{n-1}\|q_{n,i}\alpha\| = 1,$$

(8) $$p_{n,i}\|q_{n-1}\alpha\| + p_{n-1}\|q_{n,i}\alpha\| = \alpha,$$

$$\min_{0<q<q_{n+1}} \|q\alpha\| = \|q_n\alpha\|, \tag{9}$$

$$\|q_n\alpha\| = \begin{cases} 1 - \{q_n\alpha\}, & n \text{ odd}, \\ \{q_n\alpha\}, & n \text{ even}, \end{cases} \tag{10}$$

where $\|q\alpha\| = |q\alpha - p|$, $p = [q\alpha + 1/2]$. That is, $\|q\alpha\|$ is equal to the absolute difference between $q\alpha$ and its nearest integer. Note that $p_n = [q_n\alpha + 1/2]$.

Also, if $\alpha = \{0; a, a, \ldots\} = (\sqrt{a^2 + 4} - a)/2$, then

$$p_n = \frac{(1/\alpha)^n - (-\alpha)^n}{\alpha + 1/\alpha} = q_{n-1}. \tag{11}$$

3. The Three Gap Theorem

The reader is referred to van Ravenstein [18] for an account of the Three Gap Theorem as well as the proofs of many of the results used in this section. Alternatively, the reader may see van Ravenstein [19] where the theorem is also discussed with special reference to the golden number.

3.1 Order of Points

Consider N points placed in succession on a circle at an angle of α. We are interested in determining the order of the points as they appear in clockwise order on the circle. This is equivalent to ordering $(\{n\alpha\} = n\alpha \bmod 1$, $n = 0, 1, 2, \ldots, N - 1)$ into an ascending sequence. ($y \bmod x = y - x[y/x] = x\{y/x\}$.) Let $(\{u_j\alpha\})$, $j = 1, 2, \ldots, N$ be that ordered sequence. That is,

$$\{u_1, u_2, \ldots, u_N\} = \{0, 1, \ldots, N - 1\},$$

where $\{u_j\alpha\} < \{u_{j+1}\alpha\}$. It is shown in Slater [11] and Sós [14] (or see [18], Th. 2.2) that the elements u_j are obtained by the following relation,

$$u_{j+1} - u_j = \begin{cases} u_2, & 0 \le u_j < N - u_2, \\ u_2 - u_N, & N - u_2 \le u_j < u_N, \\ -u_N, & u_N \le u_j < N, \end{cases} \tag{12}$$

for $j = 1, 2, \ldots, N$, $u_1 = u_{N+1} = 0$. Points u_j and u_{j+1} delimit the j^{th} gap, which is of length $\{(u_{j+1} - u_j)\alpha\}$.

Here, we will only be concerned with the case where the circle is partitioned into gaps of just two different lengths. This occurs when $N = u_2 + u_N$ or, equivalently, when N is the denominator of a convergent to α.

It may be shown (from [18], Lemma 2.1) that, for $N = u_2 + u_N = q_{n,i}$ ($i = 1, 2, \ldots, a_n$, $n \ge 2$),

$$u_j = ((-1)^{n-1}(j - 1)q_{n-1}) \bmod q_{n,i}, \; j = 1, 2, \ldots, q_{n,i}. \tag{13}$$

For any other value of N, the circle is composed of gaps of three different lengths.

3.2 The String of Gap Types

Now let us consider the more dynamic situation—we will describe the change in gap structure induced by the addition of extra points. In particular, we are interested in the transition from a circle of q_{n-1} gaps to one of q_n gaps. Notation is needed.

Suppose the circle is partitioned into gaps of only two different lengths, say large and small. We label a large gap l and a small gap s. Let

$$\Phi_n = \phi_{n,1}\phi_{n,2} \cdots \phi_{n,q_n}$$

denote the string of gap types when $N = q_n$, ordered clockwise from the origin around the circle so that $\phi_{n,j}$ denotes the gap type (either s or l) of the j^{th} gap formed by points u_j and u_{j+1}. Assume that $\Phi_0 = s$.

For any string S and nonnegative integer t, denote by S^t the concatenation of S with itself t times, where S^0 is the empty string. For any strings S_1, S_2, we write S_1S_2 for the concatenation of S_1 followed by S_2.

Define P_n such that

$$P_n(l) = \begin{cases} s^{a_n}l, & n \text{ odd}, \\ ls^{a_n}, & n \text{ even}, \end{cases} \qquad P_n(s) = \begin{cases} s^{a_n-1}l, & n \text{ odd}, \\ ls^{a_n-1}, & n \text{ even}. \end{cases}$$

The following theorem shows that P_n is the *production rule* which describes the manner in which the string of gap types develops as more points are included on the circle. The result may, after a little effort, be derived from (12). We omit the proof, and refer the reader to Theorem 4.1 in [18].

Theorem 1:

$$\Phi_n = P_n(\Phi_{n-1}) = P_n(\phi_{n-1,1})P_n(\phi_{n-1,2}) \ldots P_n(\phi_{n-1,q_{n-1}}). \ \square$$

Example: For the golden number, $\tau = (\sqrt{5} - 1)/2$,

$$P_n(l) = \begin{cases} sl, & n \text{ odd}, \\ ls, & n \text{ even}, \end{cases} \qquad P_n(s) = l.$$

Hence,

$$\begin{aligned} \Phi_0 &= s, \\ \Phi_1 &= l, \\ \Phi_2 &= ls, \\ \Phi_3 &= sll, \\ \Phi_4 &= llsls, \\ \Phi_5 &= slsllsll. \end{aligned}$$

We now introduce the following two results which we will need to prove our main result in Section 4.2. Proposition 2 demonstrates a simple property of the production rule P_n, while Proposition 3 shows that a component of the string Φ_n is symmetric.

Let $\Theta = \theta_1\theta_2 \ldots \theta_k$ denote a string of k letters, where $\theta_i = s$ or l, $i = 1, 2, \ldots, k$. For any string S, let S^* denote the string S in reverse order. We write $P_n(\Theta)^*$ and $P_n(\theta_k)^*$ for $(P_n(\Theta))^*$ and $(P_n(\theta_k))^*$, respectively.

Proposition 2: $P_n(\Theta)^* = P_{n-1}(\Theta^*)$.

Proof:
$$\begin{aligned} P_n(\Theta)^* &= (P_n(\theta_1)P_n(\theta_2) \ldots P_n(\theta_k))^*, \\ &= P_n(\theta_k)^*P_n(\theta_{k-1})^* \ldots P_n(\theta_1)^*, \\ &= P_{n-1}(\theta_k)P_{n-1}(\theta_{k-1}) \ldots P_{n-1}(\theta_1), \\ &= P_{n-1}(\theta_k\theta_{k-1} \ldots \theta_1), \\ &= P_{n-1}(\Theta^*), \end{aligned}$$

where we have used the fact that $P_n(s)^* = P_{n-1}(s)$ and $P_n(l)^* = P_{n-1}(l)$. $\square$

Let $B_n = \phi_{n,2}\phi_{n,3} \ldots \phi_{n,q_n-1}$.

Proposition 3: $B_n^* = B_n$ $(n \geq 1)$.

Proof: For $n = 1$, the result is trivial since from Theorem 1, $\Phi_1 = s^{a_1-1}l$. Now consider the case $n \geq 2$. It is necessary to show that

$$\phi_{n,j} = \phi_{n,q_n-j+1}, \quad j = 2, 3, \ldots, q_n - 1.$$

From (13), with $i = a_n$ (since $N = q_n$),

(14) $$u_j = ((-1)^{n-1}(j - 1)q_{n-1}) \bmod q_n, \quad j = 1, 2, \ldots, q_n.$$

Thus,

$$u_{q_n-j+2} = ((-1)^{n-1}(q_n - (j - 1))q_{n-1}) \bmod q_n,$$
$$= ((-1)^n(j - 1)q_{n-1}) \bmod q_n \quad (j = 2, 3, \ldots, q_n).$$

Hence,

$$u_j + u_{q_n-j+2} = q_n \quad (j = 2, 3, \ldots, q_n).$$

Thus,

$$u_{j+1} - u_j = q_n - u_{q_n-j+1} - (q_n - u_{q_n-j+2}) \quad (j = 2, 3, \ldots, q_n - 1),$$
$$= u_{q_n-j+2} - u_{q_n-j+1},$$

from which the result follows. □

4. The Characteristic of α

4.1 General α

The following method of constructing the characteristic is described in Venkov ([20], pp. 65-68). Markoff first showed that the characteristic of α is equal to $\beta_1\beta_2\beta_3\ldots$, where

$$\beta_n = \beta_{n-1}^{a_n-1}\beta_{n-2}\beta_{n-1}, \ \beta_0 = s, \ \beta_1 = s^{a_1-1}l.$$

We mention that if α is rational, say $\alpha = \{a_0; a_1, a_2, \ldots, a_N\}$, then $\beta_1\beta_2 \ldots \beta_{N-1}(\beta_N)^\infty$ is the characteristic where N is even (so that the number of terms is odd). If N is odd, the number of terms can be made odd, as $\ldots a_{N-1}, a_N\}$ can be replaced by $\ldots a_{N-1}, a_N - 1, 1\}$, if $a_N > 1$. If $a_N = 1$ (and $\alpha \neq 1$), then $\ldots a_{N-2}, a_{N-1}, a_N\}$ can be replaced by $\ldots a_{N-2}, a_{N-1} + 1\}$.

Let $\alpha = \{0; 1, 2, 3\} = \{0; 1, 2, 2, 1\} = 7/10$. Then

$$\beta_0 = s,$$
$$\beta_1 = l,$$
$$\beta_2 = \beta_1\beta_0\beta_1 = lsl,$$
$$\beta_3 = \beta_2\beta_1\beta_2 = lsllsl,$$
$$\beta_4 = \beta_2\beta_3 = lsllsllsl.$$

The characteristic is then given by $\beta_1\beta_2\beta_3(\beta_4)^\infty$, that is,

$$llsllsllsl(lsllsllsl)^\infty.$$

Fraenkel et al. ([4], Theorem 1) offer an alternative method of construction: they show that the characteristic is equal to $\lim_{n\to\infty} \delta_n$, where

$$\delta_n = \delta_{n-1}^{a_n}\delta_{n-2}, \ \delta_0 = s, \ \delta_1 = s^{a_1-1}l. \tag{16}$$

They actually form the characteristic by means of "shift operators." It may be shown, however, that the recurrence relation (16) is an equivalent means of formulating the characteristic, in terms of the actual operations required.

Note that if $\alpha = \{a_0; a_1, a_2, \ldots, a_N\}$, then δ_N^∞ is the characteristic (if N is even).

Example: As in the above example, consider $\alpha = 7/10$. Then

$$\delta_0 = s,$$
$$\delta_1 = l,$$
$$\delta_2 = lls,$$
$$\delta_3 = llsllsl,$$
$$\delta_4 = llsllslls.$$

Thus, δ_4^∞ is the characteristic.

The method of Fraenkel et al. [4] generalizes the work done by Stolarsky ([12], Theorem 2), who shows how to generate the characteristic for the particular case where $\alpha = \{1; a, a, \ldots\}$, the positive root of $x^2 + (a - 2)x - a = 0$.

In this paper, we present a new proof of Theorem 1 in [4] and Theorem 2 in [12] for the case $\alpha \doteq \{0, a, a, \ldots\}$, the positive root of $x^2 + ax - 1 = 0$ by exhibiting a connection between the characteristic of α and its string of gap types (see Theorems 5 and 8 below).

4.2 The Characteristic of $\alpha = \{0; a, a, \ldots\}$

From now on, unless otherwise stated, assume that $\alpha = \{0; a, a, \ldots\}$. For this case, we show how the string of gap types Φ_n is generated recursively and how it is related to the characteristic of α.

Theorem 4:

$$\Phi_n = \Phi_{n-1}^{*a}\Phi_{n-2} \ (n \geq 2), \ \Phi_0 = s, \ \Phi_1 = s^{a-1}l.$$

Proof: Theorem 1 implies the truth of the assertion for $n = 2, 3$. Using the induction hypothesis we show that the result holds in general by verifying it for $n = k + 1$, assuming that it holds for $n = k$ and $n = k - 1$.

$$\begin{aligned}\Phi_{k+1} &= P_{k+1}(\Phi_k) \\ &= P_{k+1}(\Phi_{k-1}^{*a}\Phi_{k-2}) \\ &= P_{k+1}(\Phi_{k-1}^{*a})\ P_{k+1}(\Phi_{k-2}) \\ &= P_{k-1}(\Phi_{k-1}^{*})^a P_{k-1}(\Phi_{k-2}).\end{aligned}$$

Thus,

$$\Phi_{k+1} = \Phi_k^{*a}\Phi_{k-1},$$

which follows from Theorem 1 and Proposition 2 for $\Theta = \Phi_{k-1}$ and the fact that $P_{k+1} = P_{k-1}$ for all $k \geq 2$. □

The following theorem shows how the string Φ_n is related to another string Ω_n which corresponds to the first q_n elements of the characteristic. One merely places the first element of Φ_n in the penultimate position of Φ_n to obtain Ω_n.

We let $A_n = \phi_{n,1}$, B_n be as in Proposition 3, and $C_n = \phi_{n,q_n}$. Now, let

$$\Omega_n = B_n A_n C_n.$$

Theorem 5:

$$\Omega_n = \Omega_{n-1}^a\Omega_{n-2} \ (n \geq 2), \ \Omega_0 = s, \ \Omega_1 = s^{a-1}l.$$

Proof: The result is readily shown to be true for $n = 2, 3$ from direct observation of the strings Ω_2 and Ω_3. These strings derive from Φ_2 and Φ_3, which may be written down using Theorem 4. In what follows, assume that $n > 3$.

Induction on n using Theorem 1 implies

$$A_n = C_{n-1} = \begin{cases} s, & n \text{ odd}, \\ l, & n \text{ even}. \end{cases} \tag{17}$$

(This is actually true for $n \geq 2$.)

We are required to show that

$$B_nA_nC_n = (B_{n-1}A_{n-1}C_{n-1})^aB_{n-2}A_{n-2}C_{n-2},$$

or, using (17),

$$B_n = (B_{n-1}A_{n-1}C_{n-1})^aB_{n-2}. \tag{18}$$

Theorem 4 is equivalent to the statement

$$A_nB_nC_n = (C_{n-1}B_{n-1}^{*}A_{n-1})^aA_{n-2}B_{n-2}C_{n-2}.$$

Using (17) and rearranging terms leads to

$$B_n = B^*_{n-1}A_{n-1}(C_{n-1}B^*_{n-1}A_{n-1})^{a-1}A_{n-2}B_{n-2}.$$

Further manipulation gives

$$B_n = (B^*_{n-1}A_{n-1}C_{n-1})^a B_{n-2},$$

which is equivalent to (18). (Recall from Proposition 3 that $B^*_n = B_n$.) Thus, the theorem is proved. □

The following corollary gives the production rule for the string $\Omega_0\Omega_1\Omega_2\ldots$ The proof is by induction and is omitted. Note that the production rule is independent of n.

Corollary 6: Suppose that $Q(s) = s^{a-1}l$, $Q(l) = s^{a-1}ls$. Then

$$\Omega_n = Q(\Omega_{n-1}) = \Omega^a_{n-1}\Omega_{n-2} \ (n \ge 2), \ \Omega_0 = s, \ \Omega_1 = s^{a-1}l. \ \square$$

Example: For $\alpha = \tau$, we have $Q(s) = l$, $Q(l) = ls$, and $\Omega_n = \Omega_{n-1}\Omega_{n-2}$ for $n \ge 1$. Hence,

$$\begin{aligned}
\Omega_0 &= s,\\
\Omega_1 &= l,\\
\Omega_2 &= ls,\\
\Omega_3 &= lsl,\\
\Omega_4 &= lslls,\\
\Omega_5 &= lsllslsl.
\end{aligned}$$

The Golden Sequence is then $\lim_{n\to\infty}\Omega_n$. Comparing Theorem 5 with Fraenkel et al's result ([4], Theorem 1) [equivalent to our Equation (16)] identifies Ω_n as the first q_n elements of the characteristic. That is, $\Omega_n = \delta_n$, where δ_n is defined by (16). Thus, the string of gap types is generated in the same way as the characteristic, a result all the more surprising since it does not hold for all α. We proceed to verify the connection between Ω_n and the characteristic by exploiting the relationship between Φ_n and Ω_n. This, then (with Theorem 5), forms the new proof of Theorem 1 in [4] and Theorem 2 in [12] for the case $\alpha = \{0; a, a, \ldots\}$. The proof sheds light on the set of numbers for which the string of gap types corresponds to the characteristic. First, we need the following, which is proved in van Ravenstein ([17], Equation 5.12).

Lemma 7: $[k\alpha] = \left[k\dfrac{p_{n,i}}{q_{n,i}}\right]$, $k = 1, 2, \ldots, q_{n,i} - 1$ $(n \ge 2, 1 \le i \le a_n)$, where α is any irrational number. □

Let $\Omega_n = \omega_{n,1}\omega_{n,2}\ \cdots\ \omega_{n,q_n}$.

Theorem 8: For $n \ge 2$,

$$\omega_{n,j} = \begin{cases} s, & d_j = 0,\\ l, & d_j = 1,\end{cases}$$

where d_j is defined by (1) and $j = 1, 2, \ldots, q_n$.

Proof: Equation (14) is equivalent to

$$\begin{aligned}
u_j &= q_n\left\{(-1)^{n-1}(j-1)\frac{q_{n-1}}{q_n}\right\},\\
&= (-1)^{n-1}(j-1)q_{n-1} - q_n\left[(-1)^{n-1}(j-1)\frac{q_{n-1}}{q_n}\right].
\end{aligned}$$

Thus, for $j = 2, 3, \ldots, q_n - 1$,

$$\text{(19)} \qquad u_{j+1} - u_j = (-1)^{n-1}q_{n-1} - (-1)^{n-1}q_n\left(\left[j\frac{q_{n-1}}{q_n}\right] - \left[(j-1)\frac{q_{n-1}}{q_n}\right]\right),$$

$$\text{(20)} \qquad = (-1)^{n-1}q_{n-1} - (-1)^{n-1}q_n([j\alpha] - [(j-1)\alpha]).$$

The latter step follows from Lemma 7 and Equation (11). Hence, for $j = 2, 3, \ldots, q_n - 1$,

$$u_{j+1} - u_j = \begin{cases} (-1)^{n-1} q_{n-1}, & d_{j-1} = 0, \\ (-1)^{n-1}(q_{n-1} - q_n), & d_{j-1} = 1. \end{cases}$$

From (9) and (10), it may now be shown that

$$\phi_{n,j} = \begin{cases} s, & d_{j-1} = 0, \\ l, & d_{j-1} = 1, \end{cases} \tag{21}$$

where $j = 2, 3, \ldots, q_n - 1$.

To complete the proof, first note from (6) that

$$[q_n\alpha] = \begin{cases} p_n - 1, & n \text{ odd}, \\ p_n, & n \text{ even}. \end{cases}$$

From Lemma 7, $[(q_n - 1)\alpha] = [(q_n - 1)p_n/q_n] = p_n - 1$. Therefore,

$$d_{q_n - 1} = [q_n\alpha] - [(q_n - 1)\alpha] = \begin{cases} 0, & n \text{ odd}, \\ 1, & n \text{ even}. \end{cases}$$

From (14) and (9) we have

$$\phi_{n,1} = \begin{cases} s, & d_{q_n - 1} = 0, \\ l, & d_{q_n - 1} = 0. \end{cases} \tag{22}$$

The result for ϕ_{n,q_n} follows similarly. From Lemma A1 (see Appendix), $[(q_n + 1)\alpha] = p_n$. Hence,

$$d_{q_n} = [(q_n + 1)\alpha] - [q_n\alpha] = \begin{cases} 0, & n \text{ even}, \\ 1, & n \text{ odd}, \end{cases}$$

and thus, from (14) and (9),

$$\phi_{n,q_n} = \begin{cases} s, & d_{q_n} = 0, \\ l, & d_{q_n} = 1. \end{cases} \tag{23}$$

Theorem 5 and Equations (21)-(23) establish the proof. □

Corollary 9: Suppose that $\alpha = \{0; a_1, a_2, \ldots\}$, where $a_j = a_{i-j+1}$ for $j = 1, 2, \ldots, i$. Then

$$\omega_{i,j} = \begin{cases} s, & d_j = 0, \\ l, & d_j = 1. \end{cases}$$

Proof: For this value of α,

$$\frac{p_i}{q_i} = \frac{q_{i-1}}{q_i} = \{0; a_i, a_{i-1}, \ldots, a_2, a_1\}.$$

The proof is then identical to the proof of Theorem 8; in particular, the step from (19) to (20) follows. □

The correspondence between Φ_n and the characteristic does not hold for all α, as the following (counter)example shows.

Let $\alpha = \{0; 1, 2, 3, 1 + \tau\} = \dfrac{2\tau + 9}{3\tau + 13}$. Then

$$\Phi_0 = s,\ \Phi_1 = l,\ \Phi_2 = lss,\ \Phi_3 = ssslsslssl,$$

and thus,

$$\Omega_0 = s,\ \Omega_1 = l,\ \Omega_2 = sls,\ \Omega_3 = sslsslsssl,$$

which does not correspond to the characteristic, since

$$\delta_0 = s,\ \delta_1 = l,\ \delta_2 = lls,\ \delta_3 = llsllsllsl. \ \square$$

Conjecture: The correspondence between Φ_n and the characteristic holds only for α equivalent to the number $\{0; a, a, a, \ldots\}$.

APPENDIX. The Evaluation of $[N\alpha]$, $N = 1, 2, \ldots$

We have shown how one may evaluate the integer parts of positive consecutive multiples of a number by forming its characteristic. Here, we present an alternative method by which we decompose the number into terms related to its continued fraction expansion. The method appears in Fraenkel et al. [3] and is central to their paper. We offer a new and shorter proof.

Lemma A1 (see Fraenkel et al. [4], Lemma 2): Suppose that $n > 0$ and $0 < q < q_n$. Then $[(q + q_{n-1})\alpha] = p_{n-1} + [q\alpha]$.

Lemma A2 (see, e.g., Fraenkel [5], Theorem 3): There is a unique decomposition of any natural number N in the form

$$N = \sum_{i=0}^{m} b_i q_i,$$

where the b_i's are integers; $0 \le b_0 < q_1$, $0 \le b_i \le a_{i+1}$, $i > 0$, and $b_i = a_{i+1}$, only if $b_{i-1} = 0$. Since this expansion is unique,

$$\text{(A)} \qquad \sum_{i=0}^{n} b_i q_i < q_{n+1}.$$

Theorem A3: If $N = \sum_{i=k}^{m} b_i q_i$, then

$$[N\alpha] = \begin{cases} \sum_{i=k}^{m} b_i p_i, & k \text{ even}, \\ -1 + \sum_{i=k}^{m} b_i p_i, & k \text{ odd} \end{cases}$$

where $b_k \ne 0$ (i.e., $k = \max\{j : b_j > 0\}$).

Proof: If $N = \sum_{i=k}^{m} b_i q_i$, $b_k \ne 0$, then

$$[N\alpha] = \left[\left(\sum_{i=k}^{m-1} b_i q_i + b_m q_m\right)\alpha\right] = \left[\left(\sum_{i=k}^{m-1} b_i q_i + (b_m - 1)q_m + q_m\right)\alpha\right].$$

From (A),

$$\sum_{i=k}^{m-1} b_i q_i + (b_m - 1)q_m < b_m q_m \le a_{m+1} q_m < q_{m+1}.$$

Hence, from Lemma A1,

$$[N\alpha] = p_m + \left[\left(\sum_{i=k}^{m-1} b_i q_i + (b_m - 1)q_m\right)\alpha\right].$$

Further application of (A) and the lemma leads to

$$[N\alpha] = b_m p_m + \left[\sum_{i=k}^{m-1} b_i q_i \alpha\right].$$

Clearly, we are led to

$$[N\alpha] = \sum_{i=k+1}^{m} b_m p_m + [b_k q_k \alpha].$$

From (6),

$$b_k q_k \alpha - b_k p_k = \frac{b_k(-1)^k}{t_{k+1}p_k + p_{k+1}}.$$

Thus, $-1 < b_k(q_k\alpha - p_k) < 1$, since $0 \le b_k \le a_{k+1}$. Hence,

$$[b_k q_k \alpha] = \begin{cases} b_k p_k, & k \text{ even}, \\ b_k p_k - 1, & k \text{ odd}. \end{cases}$$

This completes the proof. □

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References

1. J. Bernoulli III. "Sur une Nouvelle Espece de Calcul." *Recueil pour les Astronomes 1, 2* (1772), Berlin.
2. E. B. Christoffel. "Observatio Arithmetica." *Annali Di Matematica 2nd Ser 6* (1875):148-52.
3. A. S. Fraenkel, J. Levitt, & M. Shimshoni. "Characterization of the Set of Values $f(n) = [n\alpha]$, $n = 1, 2, \ldots$." *Discrete Math. 2* (1972):335-45.
4. A. S. Fraenkel, M. Mushkin, & U. Tassa. "Determination of $[n\theta]$ by Its Sequence of Differences." *Canad. Math. Bull. 21* (1978):441-46.
5. A. S. Fraenkel. "Systems of Enumeration." *Amer. Math. Monthly 102* (1985): 105-14.
6. J. H. Halton. "The Distribution of the Sequence $\{n\xi\}$ $(n = 0, 1, 2, \ldots)$." *Proc. Camb. Phil. Soc. 71* (1965):665-70.
7. A. Y. Khintchine (tr. P. Wynn). *Continued Fractions*. Groningen: P. Noordhoff Ltd., 1963.
8. L. Kuipers & H. Neiderreiter. *Uniform Distribution of Sequences*. New York: John Wiley & Sons, 1974.
9. A. A. Markoff. "Sur une Question de Jean Bernoulli." *Math. Ann. 19* (1882): 27-36.
10. A. Revuz. "Sur la Repartition des Points $e^{vi\theta}$." *C. R. Acad. Sci. 228* (1949):1466-67.
11. N. B. Slater. "Caps and Steps for the Sequence $n\theta$ mod 1." *Proc. Camb. Phil. Soc. 73* (1967):1115-22.
12. K. B. Stolarsky. "Beatty Sequences, Continued Fractions, and Certain Shift Operators." *Canad. Math. Bull. 19* (1976):473-82.
13. V. T. Sós. "On the Theory of Diophantine Approximations. I." *Acta Math. Hung. 8* (1957):461-72.
14. V. T. Sós. "On the Distribution mod 1 of the Sequence $n\alpha$." *Ann. Univ. Sci. Budapest. Eötvos sect. Math. 1* (1958):127-34.
15. S. Swierckowski. "On Successive Settings of an Arc on the Circumference of a Circle." *Fund. Math. 46* (1958):187-89.

16. K. Tognetti, G. Winley, & T. van Ravenstein. "The Search for the Golden Sequence." Dept. of Math. Preprint 2/87, University of Wollongong.
17. T. van Ravenstein. "On the Discrepancy of the Sequence Formed from Multiples of an Irrational Number." *Bull. Austral. Math. Soc. (Series A) 31* (1985):329-38.
18. T. van Ravenstein. "The Three Gap Theorem (Steinhaus Conjecture)." *J. Austral. Math. Soc. (Series A) 45* (1988):360-70.
19. T. van Ravenstein. "Optimal Spacing of Points on a Cricle." *Fibonacci Quarterly 27.1* (1989):18-24.
20. B. A. Venkov. *Elementary Number Theory*. Groningen: Wolters-Noordhoff, 1970.

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Announcement

FOURTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

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Department of Mathematics and Computer Science

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