ON THE NUMBER OF PROPAGATION PATHS IN MULTILAYER MEDIA

Jon T. Butler
Naval Postgraduate School, Code EC/Bu, Monterey, CA 93943
(Submitted October 1988)

1. Introduction

We consider the problem of enumerating paths of the type shown in Figure 1. A wave leaves A and arrives at B along a path that is straight line except perhaps at the intersection with horizontal lines where the wave may be reflected. The layers between lines represent homogeneous media through which the wave travels in a straight line only. At the boundary between layers, a wave is either reflected or transmitted. This models, for example, sound in sea water and electromagnetic waves in soil associated with power transmission, LaGracé et al. [6]. Figure 1 shows that points A and B are embedded between two layers, in which case a path may cross the AB line some number of times. A similar problem exists in the reflection of light by adjacent panes of glass.

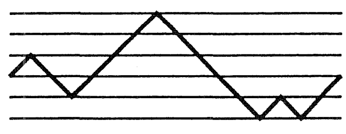

FIGURE 1. The path problem

It is known that the number of paths P_n with n reflections in two panes of glass forms a Fibonacci sequence [9, pp. 162-63, 3]. Extensions of this, including more panes and the addition of a mirror, have also been considered [2, 4, 5, 10].

Let there be $m_1 \geq 0$ layers above the AB line and $m_2 \geq 0$ below. Any path from A to B consists of an even number $2n$ of traverses across layers. We seek $N_{m_1, m_2}(n)$, the number of paths from A to B consisting of $2n$ traverses.

2. Special Cases

There are interesting special cases. The two layer model, $m_1 = 0$ and $m_2 = 2$ has been considered in electromagnetic wave propagation in soil, LaGracé et al. [7]. When $m_1, m_2 \geq n$, the path problem is equivalent to the following.

> Consider a city neighborhood that consists of n by n square blocks. How many different paths of minimum length are there from the northwest corner to the southeast corner?

View a path as an ordered sequence of $2n$ letters, n E's and n S's. The path is determined as follows. Starting from the leftmost letter, consider each letter as a specification of whether to go east or south at the current intersection. At the end of the sequence, a traveler will have gone n blocks east and n blocks south. Since there is a one-to-one correspondence between paths and sequences, the number of paths is the number of ways to choose where in the sequence the S's should go, or

$$N_{\geq n, \geq n}(n) = \binom{2n}{n}.$$

When $m_1 = 0$ and $m_2 \geq n$, the problem is equivalent to Problem 33(a) of Lovasz (see [8]):

> How many monotonic mappings of $\{1, \ldots, n\}$ into itself satisfy the condition $f(x) \le x$ for every $1 \le x \le n$?

A monotonic mapping can be represented as dots on a grid, as shown in Figure 2a. A path lying entirely on grid lines is drawn through the dots, beginning at (1, 1), point A, and ending at $(n + 1, n + 1)$, point B. If $f(x) = f(x + 1)$, the segment $(x, f(x)) \to (x + 1, f(x))$ is part of the path. If $f(x) < f(x + 1)$, the subpaths $(x, f(x)) \to (x + 1, f(x))$ and $(x + 1, f(x)) \to (x + 1, f(x + 1))$ are part of the path. Also, subpaths $(n, f(n)) \to (n + 1, f(n))$ and $(n + 1, f(n)) \to ((n + 1), (n + 1))$ are part of the path.

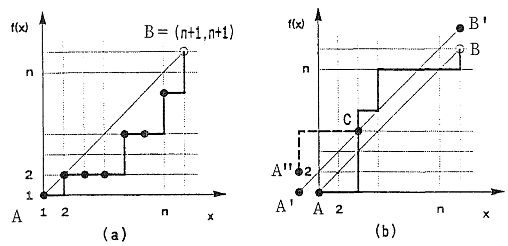

FIGURE 2. Monotonic mapping equivalence to the path problem

The restriction $f(x) \le x$ for every $1 \le x \le n$ precludes a path from crossing the AB line. It follows that the number of mappings is $N_{0,\ge n}(n)$. An interesting argument [8, p. 163], yields a simple expression for this number. Figure 2b shows two additional points (0, 1), point A', and $(n + 1, n + 2)$, point B'. All paths from A to B below the AB line that never cross it are precisely those paths from A to B which never *meet* the A'B' line. The total number of paths between A and B is $\binom{2n}{n}$, and if we subtract the number of paths which meet the A'B' line, we have our result. Figure 2b shows a path which meets the A'B' line (and also crosses it). Let C be the first point at which a path from A to B meets the A'B' line. If we reflect the segment AC about the A'B' line, we obtain the segment A"C, where A" is point (0, 2). Thus, any path A to B which *meets* the A'B' line can be converted to a path A"B. Further, the converse is true. Thus, the number of paths from A to B meeting the A'B' line is equal to the number of unrestricted paths from A" to B. This is

$$\binom{n - 1 + n + 1}{n - 1} = \binom{2n}{n - 1}.$$

It follows that,

$$N_{0,\ge n}(n) = N_{\ge n,0}(n) = \binom{2n}{n} - \binom{2n}{n - 1} = \frac{1}{n + 1}\binom{2n}{n},$$

which is a *Catalan number*. From this, it follows that the total number of paths from A to B of $2n$ traverses is reduced by $1/(1 + n)$ when no path is allowed to cross the AB line. Consider now a more general case.

3. The Number of Paths Where the Media Is on One Side Only

Let

$$F_m(x) = N_m(1)x + N_m(2)x^2 + \cdots + N_m(i)x^i + \cdots$$

be the ordinary generating function for $N_{0,m}(i)$ [$= N_{m,0}(i)$]. Then, $F^2(x)$ enumerates paths having one identified intersection with the AB line; that is, having one intersection on the AB line distinct from all other such intersections. An end point is *not* considered an intersection. In a path with p intersections with the AB line, there are $\binom{p}{1}$ ways a single identified intersection can be chosen. Thus, such a path is counted $\binom{p}{1}$ times in $F_m^2(x)$. Specifically, $F_m(x)$ enumerates the ways the path to the left of the identified point can be chosen, $F_m(x)$ enumerates the ways to the right, and $F_m^2(x)$ enumerates the ways both can be chosen. In a similar manner, $F_m^3(x)$ enumerates paths with two identified intersections on the AB line, etc. Consider

$$G_m(x) = F_m^2(x) - F_m^3(x) + F_m^4(x) - \cdots = \frac{F_m^2(x)}{1 + F_m(x)}.$$

$G_m(x)$ enumerates paths with *at least one* intersection with the AB line. Specifically, a path with *exactly* p intersections with the AB line is counted $\binom{p}{1}$ times in $F_m^2(x)$, $\binom{p}{2}$ times in $F^3(x)$, ..., and $\binom{p}{p}$ times in $F_m^{p+1}(x)$. Thus, a path with exactly p intersections is counted in $G_m(x)$ once:

$$\binom{p}{1} - \binom{p}{2} + \cdots + (-1)^{p+1}\binom{p}{p} = 1.$$

The number of paths having *no* intersection with the AB line is $N_{m-1}(n - 1)$. This is enumerated in the ordinary generating function $xF_{m-1}(x)$. Thus,

$$F_m(x) = \frac{F_m^2(x)}{1 + F_m(x)} + xF_{m-1}(x) + x,$$

where the $+x$ term is the initial condition $N_m(1) = 1$. Solving for $F_m(x)$ yields

$$F_m(x) = \frac{x}{\dfrac{1}{F_{m-1}(x) + 1} - x}. \tag{1}$$

We can solve for $F_m(x)$ iteratively. When $m = 1$, there is only one path and

$$F_1(x) = x + x^2 + x^3 + \cdots = \frac{x}{1 - x}.$$

$F_2(x)$ is obtained by substituting $x/(1 - x)$ for $F_{m-1}(x)$ [$= F_1(x)$] in (1). $F_3(x)$ and other generating functions are obtained in a similar manner. Table 1 shows the generating functions $F_m(x)$ for $1 \le m \le 5$. Also shown is the corresponding power series expansion.

Let $F_\infty(x)$ be the generating function for the number of paths when there are arbitrarily many layers below the AB line. An expression for $F_\infty(x)$ can be obtained by substituting $F_\infty(x)$ for $F_m(x)$ and $F_{m-1}(x)$ in (1). This yields an expression that is quadratic in $F_\infty(x)$, which can be solved to produce the expression shown in Table 1.

We can find closed form expressions for the approximate number of paths by a manipulation of the generating function. We illustrate using $F_3(x)$.

$$F_3(x) = x\,\frac{1 - x}{1 - 3x + x^2} = x\,\frac{\dfrac{(1 - 5^{1/2})}{(5 - 2\cdot 5^{1/2})}}{1 - x\,\dfrac{2}{3 - 5^{1/2}}} + x\,\frac{\dfrac{(1 + 5^{1/2})}{(5 + 3\cdot 5^{1/2})}}{1 - x\,\dfrac{2}{3 + 5^{1/2}}}.$$

Let $a_n \sim b_n$ mean $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 1$. Then, we can write,

$$N_3(n) \sim \frac{(1 - 5^{1/2})}{(5 - 3\cdot 5^{1/2})}\left(\frac{2}{3 - 5^{1/2}}\right)^{n-1} = 0.724(2.618)^{n-1}.$$

TABLE 1. Generating functions, power series expansion, and closed form expressions for the number of paths with $2n$ traverses when there are m layers below the AB line and none above the AB line

Generating Function $F_m(x)$	Power Series	Closed Form Expression
$F_1(x) = \frac{x}{1-x}$	$x + x^2 + x^3 + x^4 + x^5 + \cdots$	$N_1(n) = 1$
$F_2(x) = \frac{x}{1-2x}$	$x + 2x^2 + 4x^3 + 8x^4 + 16x^5 + \cdots$	$N_2(n) = 2^{n-1}$
$F_3(x) = \frac{x - x^2}{1 - 3x + x^2}$	$x + 2x^2 + 5x^3 + 13x^4 + 34x^5 + \cdots$	$N_3(n) \sim 0.724(2.618)^{n-1}$
$F_4(x) = \frac{x - 2x^2}{1 - 4x + 3x^2}$	$x + 2x^2 + 5x^3 + 14x^4 + 41x^5 + \cdots$	$N_4(n) \sim 0.5\,(3)^{n-1}$
$F_5(x) = \frac{x - 3x^2 + x^3}{1 - 5x + 6x^2 - x^3}$	$x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + \cdots$	$N_4(n) \sim 0.349(3.247)^{n-1}$
$F_\infty(x) = x(F_\infty(x)+1)^2 = \frac{1-2x-(1-4x)^{1/2}}{2x}$	$x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + \ldots + \frac{1}{n+1}\binom{2n}{n}x^n + \ldots$	$N_\infty(n) \sim (\pi n^3)^{-1/2}\,4^n = 0.564\,n^{-3/2}\,4^n$

To find an approximation of a form similar to those given earlier for

$$N_\infty(n) = \frac{1}{n+1}\binom{2n}{n},$$

we can represent $\binom{2n}{n}$ in factorials and use Stirling's approximation. Alternatively, we can apply Theorem 5 of Bender [1] to the generating function for $N_\infty(n)$. In either case, we obtain

$$N_\infty(n) \sim (\pi n^3)^{-1/2}\,4^n.$$

Table 2 shows the values of the number of paths of $2n$ traverses, where there are m layers. These entries were obtained by a program to solve for the coefficients of the various generating functions $F_m(x)$ using a symbolic mathematical manipulation package.

TABLE 2. Number of paths with $2n$ traverses when there are layers below the AB line and none above the AB line

m/n	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	4	8	16	32	64	128	256	512
3	1	2	5	13	34	89	233	610	1597	4181
4	1	2	5	14	41	122	365	1094	3281	9842
5	1	2	5	14	42	131	417	1341	4334	14041
∞	1	2	5	14	42	132	429	1430	4862	16796

4. The Number of Paths Where the Media Is on Both Sides

The calculation for $F_{m_1,\,m_2}(x) = N_{m_1,\,m_2}(1)x + N_{m_1,\,m_2}(2)x^2 + \cdots$ can be made in terms of the case just considered. Specifically,

$$F_{m_1,\,m_2}(x) = \frac{F^2_{m_1,\,m_2}(x)}{1 + F_{m_1,\,m_2}(x)} + xF_{m_1-1,\,0}(x) + xF_{0,\,m_2-1}(x) + 2x, \tag{2}$$

for $m_1, m_2 \geq 1$. $F^2_{m_1, m_2}/(1 + F_{m_1, m_2}(x))$ counts paths from A to B with at least one intersection with the AB line. $xF_{m_1 - 1, 0}(x)$ and $xF_{0, m_2 - 1}(x)$ count paths that are entirely above and below the AB line, respectively. $+2x$ represents the initial condition $N_{m_1, m_2}(1) = 2$ when $m_1, m_2 \geq 1$.

Solving (2) for $F_{m_1, m_2}(x)$ yields

$$(3) \qquad F_{m_1, m_2}(x) = \frac{x}{\dfrac{1}{F_{m_1 - 1, 0}(x) + F_{0, m_2 - 1}(x) + 2} - x}.$$

For the special case of $m_1 = m_2 = m$ we have, from (3),

$$F_{m, m}(x) = \frac{x}{\dfrac{1}{2(F_{m-1}(x) + 1)} - x}.$$

Table 3 shows the generating functions for $F_{m, m}(x)$ for $1 \leq m \leq 5$ and ∞, and Table 4 shows the number of paths $N_{\geq n, \geq n}(n)$ when there are layers above and below the AB line. These show clearly the significantly larger number of paths which exist when they are allowed to cross the AB line.

TABLE 3. Generating functions, power series expansion, and closed form expressions for the number of paths with $2n$ traverses when there are m layers above and below the AB line

Generating Function $F_{m,m}(x)$	Power Series	Closed Form Expression
$F_{1,1}(x) = \dfrac{2x}{1-2x}$	$2x + 4x^2 + 8x^3 + 16x^4 + 32x^5 + \ldots$	$N_{1,1}(n) = 2(2)^{n-1}$
$F_{2,2}(x) = \dfrac{2x}{1-3x}$	$2x + 6x^2 + 18x^3 + 54x^4 + 162x^5 + \ldots$	$N_{2,2}(n) = 2(3)^{n-1}$
$F_{3,3}(x) = \dfrac{2x-2x^2}{1-4x+2x^2}$	$2x + 6x^2 + 20x^3 + 68x^4 + 232x^5 + \ldots$	$N_{3,3}(n) \sim 1.707(3.414)^{n-1}$
$F_{4,4}(x) = \dfrac{2x-4x^2}{1-5x+5x^2}$	$2x + 6x^2 + 20x^3 + 70x^4 + 250x^5 + \ldots$	$N_{4,4}(n) \sim 1.447(3.618)^{n-1}$
$F_{5,5}(x) = \dfrac{2x-6x^2+2x^3}{1-6x+9x^2-2x^3}$	$2x + 6x^2 + 20x^3 + 70x^4 + 252x^5 + \ldots$	$N_{4,4}(n) \sim 1.244(3.732)^{n-1}$
$F_{\infty,\infty}(x) = \dfrac{2x(F_{\infty}(x)+1)}{1-2x(F_{\infty}(x)+1)} = (1-4x)^{-1/2} - 1$	$2x + 6x^2 + 20x^3 + 70x^4 + 252x^5 + \ldots + \binom{2n}{n} x^n + \ldots$	$N_{\infty,\infty}(n) \sim (\pi n)^{-1/2} 4^n = 0.564 n^{-1/2} 4^n$

TABLE 4. Number of paths with $2n$ traverses when there are layers above and below the AB line

m/n	1	2	3	4	5	6	7	8	9	10
1	2	4	8	16	32	64	128	256	512	1024
2	2	6	18	54	162	486	1458	4374	13122	39366
3	2	6	20	68	232	792	2704	9232	31520	107616
4	2	6	20	70	250	900	3250	11750	42500	153750
5	2	6	20	70	252	922	3404	12630	46988	175066
∞	2	6	20	70	252	924	3432	12870	48620	184756

Acknowledgment

The author is grateful for very helpful comments by the referee.

References

1. E. A. Bender. "Asymptotic Methods in Enumeration." *SIAM Review 16.4* (1974): 485-515.
2. I. Bruce. "Sequences Generated by Multiple-Reflections." *Fibonacci Quarterly 24.3* (1986):268-72.
3. M. Gardner. *Mathematical Circus*. New York: Vintage Books, 1981.
4. B. Junge & V. E. Hoggatt, Jr. "Polynomials Arising from Reflections Across Multiple-Plates." *Fibonacci Quarterly 11.3* (1973):285-91.
5. V. E. Hoggatt, Jr., & M. Bicknell-Johnson. "Reflections Across Two and Three Glass Plates." *Fibonacci Quarterly 17.2* (1979):118-41.
6. P. J. LaGracé, J. L. Houle, Y. Gervais, & D. Mukhedkar. "Evaluation of the Voltage Distribution around Toroidal HVDC Ground Electrodes in n-Layer Soils." IEEE 1987 Summer Power Meeting, San Francisco, CA, July 12-17, *IEEE PES Trans. Paper No. 87* (1987):521-28.
7. P. J. LaGracé, J. L. Houle, H. Greiss, & D. Mukhedkar. "Computer Aided Design of a Toroidal Ground Electrode in a Two-Layer Soil." *IEEE Trans. on Power Delivery PWRD-2 No. 3* (1987):744-49.
8. L. Lovász. *Combinatorial Problems and Exercises*. Amsterdam: North Holland Publishing, 1979.
9. L. Moser & M. Wyman. Problem B-6. *Fibonacci Quarterly 1.1* (1963):74.
10. L. Moser & M. Wyman. "Multiple Reflections." *Fibonacci Quarterly 11.3* (1973):302-06.

*****