ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by

A. P. Hillman

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should include solutions.

Anyone desiring acknowledgment of contributions should enclose a stamped, self-addressed card (or envelope).

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-676 *Proposed by Herta T. Freitag, Roanoke, VA*

Let T_n be the n^{th} triangular number $n(n + 1)/2$. Characterize the positive integers n such that

$$T_n \,\Big|\, \sum_{i=1}^{n} T_i.$$

B-677 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $T_n = n(n + 1)/2$. Characterize the positive integers n with

$$\sum_{i=1}^{n} T_i \,\Big|\, \sum_{i=1}^{n} T_i^2.$$

B-678 *Proposed by R. André-Jeannin, Sfax, Tunisia*

Show that L_{4n} and L_{4n+3} are never triangular numbers.

B-679 *Proposed by R. André-Jeannin, Sfax, Tunisia*

Express $L_{n-2}L_{n-1}L_{n+1}L_{n+2}$ as a polynomial in L_n.

B-680 *Proposed by Russell Jay Hendel & Sandra A. Monteferrante, Dowling College, Oakdale, NY*

For an integer $a \ge 0$, define a sequence $x_0, x_1, \ldots$ by $x_0 = 0$, $x_1 = 1$, and $x_{n+2} = ax_{n+1} + x_n$ for $n \ge 0$. Let $d = (a^2 + 4)^{1/2}$. For $n \ge 2$, what is the nearest integer to dx_n?

B-681 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let n be a nonnegative integer, $k \ge 2$ an even integer, and $r \in \{0, 1, \ldots, k-1\}$. Show that

$$F_{kn+r} \equiv (F_{k+r} - F_r)n + F_r \pmod{L_k - 2}.$$

SOLUTIONS

Golden Geometric Progressions

B-652 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $\alpha = (1 + \sqrt{5})/2$,

$$S_1(n) = \sum_{k=1}^{n} \alpha^k \quad \text{and} \quad S_2(n) = \sum_{k=1}^{n} \alpha^{-k}.$$

Determine m as a function of n such that $\dfrac{S_1(n)}{S_2(n)} - \alpha F_m$ is a Fibonacci number.

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY

Both $S_1(n)$ and $S_2(n)$ are geometric series, whose sums are

$$S_1(n) = \frac{\alpha(\alpha^n - 1)}{\alpha - 1} \quad \text{and} \quad S_2(n) = \frac{1}{\alpha^n} \cdot \frac{\alpha^n - 1}{\alpha - 1}.$$

respectively. Hence, if β denotes $(1 - \sqrt{5})/2$, then

$$\frac{S_1(n)}{S_2(n)} - \alpha F_m = \alpha(\alpha^n - F_m) = \frac{\alpha^2(\alpha^n - \alpha^{m-1}) + (\alpha^n - \beta^{m-1})}{\alpha - \beta} = F_n$$

when $m = n + 1$.

Also solved by R. André-Jeannin, Paul S. Bruckman, L. Cseh, Piero Filipponi, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Sahib Singh, and the proposer.

Pythagorean Triples

B-653 *Proposed by Herta T. Freitag, Roanoke, VA*

The sides of a triangle are $a = F_{2n+3}$, $b = F_{n+3}F_n$, and $c = F_3F_{n+2}F_{n+1}$, with n a positive integer.

(i) Is the triangle acute, right, or obtuse?
(ii) Express the area as a product of Fibonacci numbers.

Solution by Paul S. Bruckman, Edmonds, WA

Note that

$$b = (F_{n+2} + F_{n+1})(F_{n+2} - F_{n+1}) = F_{n+2}^2 - F_{n+1}^2;$$

$$c = 2F_{n+2}F_{n+1};$$

and $$a = F_{n+2}^2 + F_{n+1}^2.$$

We readily see that the given triangle is a Pythagorean (right) triangle, and that it satisfies: $a^2 = b^2 + c^2$, i.e., a is the hypotenuse.

If A is its area, then

$$A = \frac{1}{2}bc = F_n F_{n+1} F_{n+2} F_{n+3}.$$

Also solved by L. Cseh, Piero Filipponi, L. Kuipers, Y. H. Harris Kwong, Bob Prielipp, H.-J. Seiffert, Sahib Singh, and the proposer.

Infinite Series

B-654 *Proposed by Alejandro Necochea, Pan American U., Edinburgh, TX*

Sum the infinite series

$$\sum_{k=1}^{\infty} \frac{1 + 2^k}{2^{2k}} F_k.$$

Solution by Wray Brady, Axixic Jalisco, Mexico

$f(x) = x/(1 - x - x^2)$ is the generating function for the series

$$\sum_{k=1}^{\infty} F_k x^k,$$

which series converges if $|x| < 1/\alpha$. Thus, the sum of the series is

$$f(1/2) + f(1/4) = 26/11.$$

Also solved by R. André-Jeannin, Paul S. Bruckman, L. Cseh, Russell Euler, Piero Filipponi, Herta T. Freitag, Russell Jay Hendel, Joseph J. Kostal, L. Kuipers, Y. H. Harris Kwong, B. S. Popov, H.-J. Seiffert, Sahib Singh, and the proposer.

Farey Fractions

B-655 *Proposed by L. Kuipers, Sierre, Switzerland*

Prove that the ratio of integers x/y such that

$$\frac{F_{2n}}{F_{2n+2}} < \frac{x}{y} < \frac{F_{2n+1}}{F_{2n+3}}$$

and with smallest denominator y is $(F_{2n} + F_{2n+1})/(F_{2n+2} + F_{2n+3})$.

Solution by Sahib Singh, Clarion University, Clarion, PA

Since

$$\frac{F_{2n+1}}{F_{2n+3}} - \frac{F_{2n}}{F_{2n+2}} = \frac{1}{F_{2n+2}F_{2n+3}},$$

therefore

$$\frac{F_{2n}}{F_{2n+2}} \quad \text{and} \quad \frac{F_{2n+1}}{F_{2n+3}}$$

can be regarded as adjacent fractions of the Farey sequence of order F_{2n+3} (see Question 5 on page 173 of *An Introduction to the Theory of Numbers* by Ivan Niven and H. S. Zuckerman, 4th ed. [New York: Wiley & Sons, 1980]). Hence, by Theorem 6.4 (*Ibid.*, page 171), the desired conclusion follows.

Also solved by R. André-Jeannin, Paul S. Bruckman, B. S. Popov, and the proposer.

Closed Form

B-656 *Proposed by Richard André-Jeannin, Sfax, Tunisia*

Find a closed form for the sum

$$S_n = \sum_{k=0}^{n} w_k p^{n-k},$$

where w_n satisfies $w_n = pw_{n-1} - qw_{n-2}$ for n in $\{2, 3, \ldots\}$, with p and q nonzero constants.

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY

It is a routine exercise to show that

$$w_k = \frac{a\alpha^k + b\beta^k}{\alpha - \beta},$$

where

$$\alpha = (p + \sqrt{p^2 - 4q})/2, \quad \beta = (p - \sqrt{p^2 - 4q})/2,$$
$$a = w_1 - \beta w_0, \text{ and } b = \alpha w_0 - w_1.$$

The formula for w_k leads to

$$S_n = \sum_{k=0}^{n} w_k p^{n-k} = \frac{a}{\alpha - \beta}\cdot\frac{p^{n+1} - \alpha^{n+1}}{p - \alpha} + \frac{b}{\alpha - \beta}\cdot\frac{p^{n+1} - \beta^{n+1}}{p - \beta}.$$

Since $\alpha + \beta = p$ and $\alpha\beta = q$, we have

$$S_n = \frac{a\alpha(p^{n+1} - \alpha^{n+1}) + b\beta(p^{n+1} - \beta^{n+1})}{(\alpha - \beta)\alpha\beta}$$

$$= \frac{p^{n+1}(a\alpha + b\beta) - (a\alpha^{n+2} + b\beta^{n+2})}{q(\alpha - \beta)} = \frac{p^{n+1}w_1 - w_{n+2}}{q}.$$

Also solved by Paul S. Bruckman, L. Cseh, Russell Euler, Piero Filipponi, L. Kuipers, B. S. Popov, H.-J. Seiffert, and the proposer.

Disjoint Increasing Sequences

B-657 *Proposed by Clark Kimberling, U. of Evansville, Evansville, IN*

Let m be an integer and $m \geq 3$. Prove that no two of the integers

$$k(mF_n + F_{n-1}) \text{ for } k = 1, 2, \ldots, m-1 \text{ and } n = 0, 1, 2, \ldots$$

are equal. Here $F_{-1} = 1$.

Composite of solutions by Paul S. Bruckman, Edmonds, WA, and Philip L. Mana, Albuquerque, NM

Assume that $m \geq 3$; $u, v \in N = \{0, 1, \ldots\}$;

(1) $\quad j, k \in \{1, 2, \ldots, m-1\}$;

(2) $\quad j(mF_u + F_{u-1}) = k(mF_v + F_{v-1})$.

We wish to show that $j = k$ and $u = v$. It is easily seen that

$$mF_n + F_{n-1} > 0, \text{ for } n \geq 0.$$

Therefore, if $u = v$, then $j = k$ as desired. Now there is no loss of generality in assuming that $0 \leq u < v$.

If $u = 0$, then $v > 0$ and (2) gives

$$j = k(mF_v + F_{v-1}) \geq m,$$

which contradicts (1). If $u = 1$, then $v > 1$, and (2) gives

$$mj = k(mF_v + F_{v-1}).$$

Thus, $m(j - kF_v) = kF_{v-1} \geq 1$. So

$$j - kF_v \geq 1 \quad \text{and} \quad j > kF_v \geq kF_{v-1} = m(j - kF_v) \geq m,$$

again contradicting (1).

Now we can assume that $2 \leq u < v$. Also, we assume that $\gcd(j, k) = 1$ since this is the situation when j and k are divided by $\gcd(j, k)$ in (1). Then (2) shows that

$$j \mid (mF_v + F_{v-1})$$

and we let $mF_v + F_{v-1} = cj$. This leads to $mF_u + F_{u-1} = ck$ and

$$c(kF_v - jF_u) = (mF_u + F_{u-1})F_v - (mF_v + F_{v-1})F_u$$
$$= F_{u-1}F_v - F_{v-1}F_u = (-1)^u F_{v-u}.$$

Hence, $c \mid F_{v-u}$, and we let $d = F_{v-u}/c$. Now $kF_v - jF_u = (-1)^u d$; therefore,

(3) $\quad jF_u = kF_v - (-1)^u d = [(mF_u + F_{u-1})/c]F_v - (-1)^u d.$

Since $v \geq 3$ and $v - u < v$, we have $F_v > F_{v-u} = cd$. Hence, $u \geq 2$, and (3) gives

$$jF_u > (mF_u + F_{u-1})d - d \geq (mF_u + 1)d - d = mdF_u.$$

Thus, $j > md \geq m$. This contradiction and the previous work show that $u = v$ and $j = k$.

Also solved by Piero Filipponi and the proposer.

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