GENERALIZED FIBONACCI SEQUENCES VIA ARITHMETICAL FUNCTIONS

P. J. McCarthy
University of Kansas, Lawrence, KS 66045

R. Sivaramakrishnan
University of Calicut, Calicut 673 635, India
(Submitted December 1988)

1. Introduction

Haukkanen has pointed out in [3] the connection that exists between the specially multiplicative arithmetical functions (to be defined in Section 2) and the Fibonacci sequence. In this paper we shall discuss the similar connection that exists between certain arithmetical functions and the generalized sequences $\{w_n\}$, where $w_n = w_n(a, b; p, q)$, i.e.,

$$w_0 = a, \; w_1 = b, \text{ and } w_n = pw_{n-1} - qw_{n-2}, \text{ for } n \geq 2,$$

which have been studied by Horadam [5], [6], [7], and others (see, for example, [10]). Here, a, b, p, and q are arbitrary complex numbers, except that $a \neq 0$.

Our aim is to characterize the family of generalized sequences in terms of a family of arithmetical functions, and to illustrate how certain properties of the sequences reflect properties of the arithmetical functions. This work was done while the second author was a visiting Stouffer professor at the University of Kansas during the 1987-1988 academic year.

General background material on arithmetical functions can be found in most texts on number theory, and more specialized material is in the books by Apostol [1] and McCarthy [14]. We shall review here and in the following section several concepts which are used in this paper.

A (complex-valued) arithmetical function, f, is called multiplicative if $f(1) = 1$ and $f(rs) = f(r)f(s)$ whenever $(r, s) = 1$: it is called completely multiplicative if $f(1) = 1$ and $f(rs) = f(r)f(s)$ for all positive integers r and s. If f is an arithmetical function and t is a prime, then the formal power series

$$f_{(t)}(x) = f(1) + f(t)x + f(t^2)x^2 + \cdots$$

is called the Bell series of f at t. Bell series are discussed on pages 42-45 of [1], and in several exercises (1.97-1.102) of [14]. If f is multiplicative, then f is determined completely by its Bell series (at all primes t). If f is completely multiplicative, its Bell series at t is

$$f_{(t)}(x) = 1 + f(t)x + f(t)^2x^2 + \cdots = \frac{1}{1 - f(t)x}.$$

We shall abuse the language and refer to the closed form of the Bell series as the Bell series itself. It is the relation between arithmetical functions and their Bell series that allows us to make the connection between arithmetical functions and generalized sequences.

2. The Sequences $\{w_n\}$

An arithmetical function, f, is called specially multiplicative if there exist completely multiplicative functions g_1 and g_2 such that $f = g_1 * g_2$, the Dirichlet convolution of g_1 and g_2, i.e.,

$$f(r) = \sum_{d|r} g_1(d)g_2(r/d),$$

for all positive integers r, where d runs over all of the positive divisors of r. Specially multiplicative functions arise naturally in several contexts in number theory. However, we emphasize that examples can be constructed in a completely arbitrary manner, as follows. For each prime t, let α_t and β_t be complex numbers. Let g_1 and g_2 be the completely multiplicative functions such that, for each prime t, $g_1(t) = \alpha_t$ and $g_2(t) = \beta_t$. Let $f = g_1 * g_2$. Then f is specially multiplicative and, for each prime t and $n \geq 1$,

$$f(t^n) = \sum_{j=0}^{n} \alpha_t^j \beta_t^{n-j}.$$

Specially multiplicative functions were studied first by Vaidyanathaswamy [20] under the name "quadratic functions," and the name "specially multiplicative functions" was given to them by Lehmer [11]. These functions are discussed on pages 18-27 and 65-68 of [14] and in papers by Kesava Menon [8], McCarthy [12], [13], Mercier [15], Ramanathan [16], Rankin [17], Redmond & Sivaramakrishnan [18], and Sivaramakrishnan [19].

If f is specially multiplicative, the Bell series of f at a prime t is given by

$$f_{(t)}(x) = \frac{1}{1 - f(t)x + B(t)x^2},$$

where B is the completely multiplicative function for which $B(t) = g_1(t)g_2(t)$ for each t: we note that $f(t) = g_1(t) + g_2(t)$. Furthermore, if f is a multiplicative function such that, for each prime t, its Bell series at t is given by

$$f_{(t)}(x) = \frac{1}{1 - c_t x + d_t x^2},$$

for some complex numbers c_t and d_t, then f is specially multiplicative, as it was described earlier in this section, with α_t and β_t the (possibly equal) roots of $X^2 - c_t X + d_t$.

In [9], Lahiri defined an arithmetical function f to be quasimultiplicative if $f(1) \neq 0$ and if there is a complex number $k \neq 0$ such that $f(r)f(s) = kf(rs)$ whenever $(r, s) = 1$. It follows immediately that $k = f(1)$ and $k^{-1}f$ is multiplicative. In fact, an arithmetical function f with $f(1) \neq 0$ is quasimultiplicative if and only if $f(1)^{-1}f$ is multiplicative.

Now we can make precise the connection between the generalized sequences $\{w_n\}$ and certain arithmetical functions.

Theorem 1: For a sequence of complex numbers $\{c_n\}$, $n \geq 0$, there exist complex numbers a, b, p, and q such that $c_n = w_n(a, b; p, q)$ for all $n \geq 0$ if and only if there is a quasimultiplicative function f and a prime t such that

(i) $f(1)^{-1}f = g_1 * \mu g_2$, where g_1 is specially multiplicative, g_2 is completely multiplicative, and μ is the Möbius function, and
(ii) $c_n = f(t^n)$ for all $n \geq 0$.

Proof: The generating function of the generalized sequence $\{w_n\}$, where $w_n = w_n(a, b; p, q)$ is, from [6],

$$\sum_{n=0}^{\infty} w_n x^n = \frac{a + (b - pa)x}{1 - px + qx^2}.$$

Let t be an arbitrary prime, and let g_1 be a specially multiplicative function such that

$$g_{1(t)}(x) = \frac{1}{1 - px + qx^2},$$

and let g_1 be a completely multiplicative function such that $g_2(t) = (pa - b)/a$, so that

$$g_{2(t)}(x) = \frac{a}{a + (b - pa)x}.$$

The inverse g_2^{-1} of g_2 with respect to Dirichlet convolution is μg_2 (see Prop. 1.8 in [14]), and $g_{2(t)}^{-1}(x) = (g_{2(t)}(x))^{-1}$ (see Th. 2.25 in [1]). Therefore, if f is a quasimultiplicative function given by $f(r) = a(g_1 * \mu g_2)(r)$ for all r, when $w_n = f(t^n)$ for all $n \geq 0$.

Conversely, let f be a quasimultiplicative function for which (i) holds, and suppose that, for some prime t,

$$c_n = f(t^n) \text{ for all } n \geq 0.$$

Then $c_n = w_n(a, b; p, q)$ for all $n \geq 0$, where

$$a = f(1),\ p = g_1(t),\ b = a(g_1(t) - g_2(t)), \text{ and } q = h_1(t)h_2(t),$$

where h_1 and h_2 are completely multiplicative functions such that $g_1 = h_1 * h_2$.

3. Some Examples

Horadam pointed out in [7] that several sequences of general interest are of the kind considered in Section 2.

(A) $w_n = w_n(1, 2; 2, 1)$. $\{w_n\}$ is the sequence of positive integers. The quasimultiplicative function is τ, where $\tau(r)$ is the number of divisors of r.

(B) $w_n = w_n(1, 3; 2, 1)$. $\{w_n\}$ is the sequence of odd positive integers. The function is $\tau * \mu\lambda$, where λ is Liouville's function (see [14], p. 45).

(C) $w_n = w_n(a, a + d; 2, 1)$. $\{w_n\}$ is the arithmetical progression

$$a,\ a + d,\ a + 2d,\ \ldots .$$

The function is $a(\tau * \mu g)$, where g is the completely multiplicative function with $g(t) = 1 - d/a$. Here, and in other examples, t is an arbitrary prime.

(D) $w_n = w_n(a, aq; q + 1, q)$. $\{w_n\}$ is the geometric progression

$$a,\ aq,\ aq^2,\ \ldots .$$

The function is ah, where h is the completely multiplicative function with $h(t) = q$.

(E) The Fermat sequences $\{u_n\}$ and $\{v_n\}$, where

$$u_n = w_n(1, 3; 3, 2) = 2^{n+1} - 1 \text{ and } v_n = w_n(2, 3; 3, 2) = 2^n + 1.$$

The functions are, respectively, $h_1 * h_2$ and $2(h_1 * h_2 * \mu g_2)$, where h_1, h_2, and g_2 are completely multiplicative functions with

$$h_1(t) = 1,\ h_2(t) = 2, \text{ and } g_2(t) = 3/2.$$

(F) The Pell sequences $\{u_n\}$ and $\{v_n\}$, where

$$u_n = w_n(1, 2; 2, -1) \text{ and } v_n = w_n(2, 2; 2, -1).$$

The functions are, respectively, $h_1 * h_2$ and $2(h_1 * h_2 * \mu g_2)$, where h_1, h_2, and g_2 are completely multiplicative functions with

$$h_1(t) = 1 + \sqrt{2},\ h_2(t) = 1 - \sqrt{2}, \text{ and } g_2(t) = 1.$$

One more example. In [4], Horadam considered the sequence $\{w_n\}$, where $w_n = w_n(r,\ r + s;\ 1,\ -1)$. The function is $r(h_1 * h_2 * \mu g_2)$, where h_1, h_2, and g_2 are completely multiplicative functions with

$$h_1(t) = (1 + \sqrt{5})/2, \quad h_2(t) = (1 - \sqrt{5})/2, \text{ and } g_2(t) = s/r.$$

With $r = 1$ and $s = 0$ this is, of course, the Fibonacci sequence.

In several of the examples, $a = 1$ and $b = p$. Sequences for which this is true are of special interest, and they will be discussed in the following section. Thus, we shall consider sequences $\{u_n\}$, where

$$u_n = u_n(p,\ q) = w_n(1,\ p;\ p,\ q).$$

These are the sequences for which the corresponding arithmetical functions are specially multiplicative.

4. The Sequences $\{u_n\}$

There exist various characterizations of specially multiplicative functions, and each of them furnishes us with a characterization of the class of sequences $\{u_n\}$. Thus, we have the following theorem; no proof will be given, and the reader is referred instead to Theorem 1.12 and Exercises 1.101 and 1.102 in [14].

Theorem 2: For a sequence of complex numbers $\{c_n\}$, $n \geq 0$, the following statements are equivalent:

(i) $c_n = u_n(p,\ q)$ for complex numbers p and q, and all $n \geq 0$.

(ii) There is a specially multiplicative function f and a prime t such that $c_n = f(t^n)$ for all $n \geq 0$.

(iii) $c_0 = 1$ and there is a complex number a such that, for all $m,\ n \geq 1$,

$$c_{m+n} = c_m c_n - a c_{m-1} c_{n-1}.$$

(iv) $c_0 = 1$ and there is a complex number b such that, for all $m,\ n \geq 0$ with $m \leq n$,

$$c_m c_n = \sum_{i=0}^{m} c_{m+n-2i} b^i.$$

(v) There are complex numbers d and e such that

$$\sum_{n=0}^{\infty} c_n x^n = \frac{1}{1 - dx + ex^2}.$$

From the details of the proof of this theorem it emerges that, if (i)-(v) hold for a sequence $\{c_n\}$, then $d = p = f(t)$ and $a = b = e = q = f(t)^2 - f(t^2)$.

Results about specially multiplicative functions now yield results about the sequences $\{u_n\}$, and vice versa, of course. For example, by Theorem 2 of [18], if $u_n = u_n(p,\ q)$ for $n \geq 0$, then, for all $n \geq 1$,

$$u_n = \sum_{j=0}^{[n/2]} (-1)^j \binom{n-j}{j} p^{n-2j} q^j.$$

This is an old result about these sequences. The original reference is given on page 394 of [2].

The identities of (iii)and (iv) are special cases of the same general identity. The latter is obtained from an arithmetical identity involving specially multiplicative functions given first in [13] (see also Ex. 1.79 in [14]). Let $u_n = u_n(p,\ q)$ for $n \geq 0$. If g is an arbitrary arithmetical function and if $G = g * \mu$, and if t is any prime, then for all $m,\ n \geq 0$ with $m \leq n$,

$$(1) \qquad \sum_{i=0}^{m} G(t^i)q^i u_{m-i}u_{n-i} = \sum_{i=0}^{m} g(t^i)q^i u_{m+n-2i}.$$

If $g = \zeta$, where $\zeta(r) = 1$ for all r, then $G = \delta$, where $\delta(1) = 1$ and $\delta(r) = 0$ for all $r > 1$, and (1) is the identity of (iv). If $g = \delta$, then $G = \mu$, and (1) is the identity of (iii). If $g = \tau = \zeta * \zeta$, then $G = \zeta$, and we obtain from (1) the identity

$$\sum_{i=0}^{m} q^i u_{m-i}u_{n-i} = \sum_{i=0}^{m} (i + 1)q^i u_{m+n-2i};$$

in particular, with $m = n$,

$$\sum_{i=0}^{n} q^{n-i}u_i^2 = \sum_{i=0}^{n} (n - i + 1)q^{n-i}u_{2i}.$$

Kesava Menon [8] associated with a multiplicative function f another multiplicative function f^*, which he called the norm of f. The definition of f^* can be found on page 50 of [14] and in several of the papers in our list of references. For our purposes, it suffices to note that if g and h are completely multiplicative functions, and if $f = g * h$, then f^* is also specially multiplicative and, in fact, $f^* = g^2 * h^2$. Thus, if the sequence $\{u_n\}$, where $u_n = u_n(p, q)$, is given by $u_n = f(t^n)$ for a prime t, then we can associate with $\{u_n\}$ the sequence $\{u_n^*\}$, where $u_n^* = f^*(t^n)$. We have $u_n^* = u_n(p^*, q^*)$, where

$$p^* = f^*(t) = g(t)^2 + h(t)^2 = p^2 - 2q \text{ and } q^* = g(t)^2h(t)^2 = q^2.$$

Thus,

$$u_n^* = u_n(p^2 - 2q, q^2).$$

From Theorems 4.1 and 4.2 of Sivaramakrishnan [19], which relate the functions f and f^*, we obtain two identities relating the sequences $\{u_n\}$ and $\{u_n^*\}$:

$$(2) \qquad u_n^2 = u_n^* + 2\sum_{i=1}^{n} q^i u_{n-i}^*$$

and

$$(3) \qquad \sum_{i=0}^{n} (-1)^i u_i^2 u_{n-i}^2 = \sum_{i=0}^{n} (-1)^i u_i^* u_{n-i}^*.$$

5. Generating Functions

We can obtain some information about the generating functions of the sequences $\{u_n\}$, and related sequences, from the Dirichlet series generating functions of corresponding arithmetical functions. In this section we assume that the reader is familiar with at least some of the material in Chapter 5 of [14] on Dirichlet series. Theorems about Dirichlet series generating functions involve hypotheses concerning the convergence of the series: we shall assume that whatever convergence is required does hold.

It will suffice to give several examples. Mercier ([15], Th. 3) gave the generating function of the product of two specially multiplicative functions, and we shall use the form of his result given on page 104 of his paper. From Mercier's result we obtain the following: if $u_n = u_n(p, q)$ and $u_n' = u_n(p', q')$ for all $n \geq 0$, then

$$\sum_{n=0}^{\infty} u_n u_n' x^n = \frac{1 - qq'x^2}{1 - pp'x + [(p^2 - q)q' + (p'^2 - q')q]x^2 - pp'qq'x^3 + q^2q'^2x^4}.$$

In particular,

$$\sum_{n=0}^{\infty} u_n u_n^* x^n = \frac{1 - q^3x^2}{1 - p(p^2 - 2q)x + (p^4 - 3p^2q + 2q^2)qx^2 - p(p^2 - 2q)q^3x^3 + q^6x^4}$$

and

$$\sum_{n=0}^{\infty} u_n^2 x^n = \frac{1 - q^2x^2}{1 - p^2x + 2(p^2 - q)qx^2 - p^2q^2x^3 + q^4x^4}. \tag{4}$$

The denominator on the right-hand side of (4) factors into the product of two quadratics, one of which is $(1 - qx)^2$. Hence,

$$\sum_{n=0}^{\infty} u_n^2 x^n = \frac{1 + qx}{(1 - (p^2 - 2q)x + q^2x^2)(1 - qx)}, \tag{5}$$

and we note that

$$\sum_{n=0}^{\infty} u_n^* x^n = \frac{1}{1 - (p^2 - 2q)x + q^2x^2}. \tag{6}$$

The generating function (6) can be obtained also from the Corollary to Theorem 7 of Redmond & Sivaramakrishnan [18].

From (5) and (6), we have

$$\sum_{n=0}^{\infty} u_n^2 x^n = \frac{1 + qx}{1 - qx} \sum_{n=0}^{\infty} u_n^* x^n. \tag{7}$$

Now,

$$\frac{1 + qx}{1 - qx} = 1 + \sum_{n=1}^{\infty} 2q^n x^n.$$

Thus, if we multiply out the right-hand side of (7) and compare coefficients of x^n, we obtain (2). If we replace x by $-x$ in (7) and multiply the left- and right-hand sides of the resulting equation by the left- and right-hand sides, respectively, of (7), and then compare coefficients of x^n, we obtain (3).

From the Corollary to Theorem 8 of Redmond & Sivaramakrishnan [18], we have

$$\sum_{n=0}^{\infty} u_{2n} x^n = (1 + qx) \sum_{n=0}^{\infty} u_n^* x^n .$$

Combining this with (7) gives

$$\sum_{n=0}^{\infty} u_n^2 x^n = \left(\sum_{n=0}^{\infty} q^n x^n\right)\left(\sum_{n=0}^{\infty} u_{2n} x^n\right),$$

and if we multiply this out and compare coefficients of x^n, we obtain

$$u_n^2 = \sum_{i=0}^{n} q^{n-i} u_{2i}.$$

From Theorem 9 of the same paper, we see that, for a fixed $m \geq 1$,

$$\sum_{n=0}^{\infty} u_{m+n} x^n = (u_m - qu_{m-1}x) \sum_{n=0}^{\infty} u_n x^n. \tag{8}$$

If we multiply out the right-hand side and compare coefficients of x^n, we find that (8) is simply the expression in series form of the identity in (iii) of Theorem 2.

6. Linear Combinations of Sequences

In this section we shall obtain a result suggested by a theorem of Rankin on specially multiplicative functions ([17], Th. 5). We shall say that sequences $\{a_n^{(j)}\}$, $j = 1, \ldots, k$ are *linearly independent* if the only complex numbers $c_1, \ldots, c_k$ for which

$$c_1 a_n^{(1)} + \cdots + c_k a_n^{(k)} = 0, \text{ for all } n \geq 0,$$

for all $n \geq 0$ are $c_1 = \cdots = c_k = 0$.

Theorem 3: If $p_1, \ldots, p_k$ are distinct complex numbers and if $u_n^{(j)} = u_n(p_j, q)$, for $j = 1, \ldots, k$, then the sequences $\{u_n^{(j)}\}$ are linearly independent.

Proof: Suppose that $c_1 u_n^{(1)} + \cdots + c_k u_n^{(k)} = 0$, for $n \geq 0$. Then the first k of these equations form a system of k linear equations with $c_1, \ldots, c_k$ as the unknowns, and the matrix of coefficients is $[u_i^{(j)}]$, where $i = 0, 1, \ldots, k - 1$ and $j = 1, \ldots, k$. Its first row is 1, 1, ..., 1 and its second row is $p_1, p_2, \ldots, p_k$. Furthermore, as we have noted in Section 4,

$$u_i^{(j)} = \sum_{r=0}^{[i/2]} (-1)^r \binom{i - r}{r} p_j^{i-2r} q^r.$$

Thus, if $i \geq 2$, then by adding appropriate multiples of rows $i - 2[i/2], \ldots, i - 2$ to row i, the matrix can be transformed into one having $p_1^i, \ldots, p_k^i$ for its i^{th} row. The determinant of the matrix of coefficients is unchanged by this transformation. Thus,

$$\det[u_i^{(j)}] = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ p_1 & p_2 & \cdots & p_k \\ \vdots & \vdots & & \vdots \\ p_1^{k-1} & p_2^{k-1} & & p_k^{k-1} \end{vmatrix} = \prod_{1 \leq i < j \leq k} (p_j - p_i) \neq 0.$$

Therefore, $c_1 = \cdots = c_k = 0$.

Theorem 4: Let $p_1, \ldots, p_k$ be distinct complex numbers and let $u_n^{(j)} = u_n(p_j, q)$ for $j = 1, \ldots, k$. If, for complex numbers $c_1, \ldots, c_k$, we have

$$u_n = c_1 u_n^{(1)} + \cdots + c_k u_n^{(k)} = u_n(p, q),$$

for some p and for all $n \geq 0$, then for some h with $1 \leq h \leq k$, we have $c_h = 1$, $c_j = 0$, for $j \neq h$ and $p = p_h$.

Proof: We shall use identity (iv) of Theorem 2. We have, for all $m, n \geq 0$ with $m \leq n$,

$$\begin{aligned} u_m u_n &= \sum_{i=0}^{m} u_{m+n-2i} q^i = \sum_{i=0}^{m} q^i \sum_{j=1}^{k} c_j u_{m+n-2i}^{(j)} \\ &= \sum_{j=1}^{k} c_j \sum_{i=0}^{m} u_{m+n-2i}^{(j)} q^i = \sum_{j=1}^{k} c_j u_m^{(j)} u_n^{(j)}. \end{aligned}$$

Also,

$$u_m u_n = \sum_{j=1}^{k} c_j u_m^{(j)} u_n = \sum_{j=1}^{k} c_j u_m u_n^{(j)}.$$

Thus, if $m \leq n$,

$$\sum_{j=1}^{k} c_j (u_n - u_n^{(j)}) u_m^{(j)} = 0, \tag{9}$$

and also

$$\sum_{j=1}^{k} c_j (u_m - u_m^{(j)}) u_n^{(j)} = 0.$$

Therefore, (9) holds for all $m, n \geq 0$, without regard to the relative sizes of m and n. Hence, for each (fixed) n,

$$c_j (u_n - u_n^{(j)}) = 0, \text{ for } j = 1, \ldots, k.$$

Since $u_0 = 1$, we must have $c_j \neq 0$ for some j. Suppose $c_h \neq 0$; then $u_n = u_n^{(h)}$ for all $n \geq 0$, and

$$\sum_{\substack{j=1 \\ j \neq h}} c_j u_n^{(j)} + (c_h - 1) u_n^{(h)} = 0, \text{ for all } n \geq 0.$$

Thus, $c_h = 1$ and $c_j = 0$, for $j \neq h$. Further,

$$p_h = u_1^{(h)} = u_1 = p.$$

References

1. T. M. Apostol. *Introduction to Analytic Number Theory*. New York: Springer-Verlag, 1976.
2. L. E. Dickson. *History of the Theory of Numbers*. Vol. I. New York: Chelsea Publishing Co., 1959.
3. P. Haukkanen. "A Note on Specially Multiplicative Arithmetic Functions." *Fibonacci Quarterly* 26 (1988):325-27.
4. A. F. Horadam. "A Generalized Fibonacci Seuqence." *Amer. Math. Monthly 68* (1961):455-59.
5. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *Fibonacci Quarterly* 3 (1965):161-77.
6. A. F. Horadam. "Generating Functions for Powers of Certain Generalized Sequences of Numbers." *Duke Math. J.* 32 (1965):435-46.
7. A. F. Horadam. "Special Properties of the Sequence $W_n(a, b; p, q)$." *Fibonacci Quarterly* 5 (1967):424-34.
8. P. Kesava Menon. "Series Associated with Ramanujan's Function $\tau(n)$." *J. Indian Math. Soc.* 27 (1963):57-65.
9. D. B. Lahiri. "Hypo-Multiplicative Number-Theoretic Functions." *Aequationes Math.* 9 (1973):184-92.
10. Jin-Zei Lee & Jia-Sheng Lee. "Some Properties of the Sequence $\{W_n(a, b; p, q)\}$." *Fibonacci Quarterly* 25 (1987):111-17.
11. D. H. Lehmer. "Some Functions of Ramanujan." *Math. Student* 27 (1959):105-16.
12. P. J. McCarthy. "Busche-Ramanujan Identities." *Amer. Math. Monthly 67* (1960):966-70.
13. P. J. McCarthy. "Some More Remarks on Arithmetical Identities." *Portugal. Math.* 21 (1962):45-57.
14. P. J. McCarthy. *Introduction to Arithmetical Functions*. New York: Springer-Verlag, 1986.
15. A. Mercier. "Remarques sur les Fonctions Spécialement Multiplictives." *Ann. Sci. Math. Québec* 6 (1982):99-107.
16. K. G. Ramanathan. "Multiplicative Arithmetic Functions." *J. Indian Math. Soc.* 7 (1943):111-16.
17. R. A. Rankin. "Multiplicative Functions and Operators of Hecke Type." *Acta Math. Acad. Sci. Hungar.* 13 (1962):81-89.
18. D. Redmond & R. Sivaramakrishnan. "Some Properties of Specially Multiplicative Functions." *J. Number Theory* 13 (1981):210-27.
19. R. Sivaramakrishnan. "On a Class of Multiplicative Arithmetic Functions." *J. Reine Angew. Math.* 280 (1976):157-62.
20. R. Vaidyanathaswamy. "The Theory of Multiplicative Arithmetic Functions." *Trans. Amer. Math. Soc.* 33 (1931):579-662.

*****