GENERALIZED STAGGERED SUMS

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1. Introduction

Wiliam [8] showed that, for the recurring sequence defined by $u_1 = 0$, $u_2 = 1$, and

(1.1) $u_{n+2} = au_n + bu_{n+1}$,

(1.2) $\sum_{n=1}^{\infty} u_n/10^n = 1/(100 - 10b - a)$,

where $(b + a)/20$ and $(b - a)/20$ are less than 1 and $b = \sqrt{b^2 + 4a}$ (cf. [8]). Thus, for the Fibonacci numbers defined by the same initial conditions and $a = b = 1$, we get the "staggered sum" of Wiliam:

(1.3) $.0 + .01 + .001 + .0002 + .00003 + \cdots = 1/89.$

It is the purpose of this note to generalize the result for arbitrary-order recurring sequences, and to relate it to an arithmetic function of Atanassov [1].

2. Arbitrary-Order Sequence

More generally, for the linear recursive sequence of order k, defined by the recurrence relation

(2.1) $u_n = \sum_{j=1}^{k} (-1)^{j+1} P_j u_{n-j}, \; n > 1,$

where the P_j are integers, and with initial conditions $u_0 = 1$ and $u_n = 0$ for $n < 0$, we can establish that the formal generating function is given by

(2.2) $\sum_{n=0}^{\infty} u_n x^n = (x^k f(1/x))^{-1},$

where $f(x)$ denotes the auxiliary polynomial

(2.3) $f(x) = x^k + \sum_{j=1}^{k} (-1)^j P_j x^{k-j}.$

Proof: If $u(x) = u_0 + u_1 x + u_2 x^2 + \cdots + u_k x^k + \cdots,$

then $-P_1 x u(x) = -P_1 u_0 x - P_1 u_1 x^2 - \cdots - P_1 u_{k-1} x^k - \cdots,$

and $(-1)^k x^k P_k u(x) = (-1)^k P_k u_0 x^k + \cdots,$

so that

$$u(x)\left(1 + \sum_{j=1}^{k} (-1)^j P_j x^j\right) = u_0 \quad \text{or} \quad u(x)x^k\left(x^{-k} + \sum_{j=1}^{k} (-1) P_j x^{j-k}\right) = 1$$

or

$$u(x)x^k f(1/x) = 1.$$

We see then that, for $k = 2$ and $P_1 = -P_2 = 1$, we get Wiliam's case in which $x = 1/10$, namely

$$\sum_{n=0}^{\infty} u_n/10^n = 1/10^{-2}f(10) = 1/\frac{1}{100}(100 - 10b - a),$$

or

$$\sum_{n=0}^{\infty} u_n/10^{2+n} = 1/(100 - 10b - a)$$

(where his initial values are displaced by 2 from those here).

3. Atanassov's Arithmetic Functions

Atanassov [1] has defined arithmetic functions ϕ and Ψ as follows. For

$$n = \sum_{i=1}^{j} a_i 10^{j-i}, \quad a_i \in \mathbb{N},$$
$$\equiv a_1 a_2 \ldots a_j, \quad 0 \le a_i \le 9,$$

let $\phi\colon \mathbb{N} \to \mathbb{N}$ be defined by

$$\phi(n) = \begin{cases} 0 & \text{for } n = 0, \\ \sum_{i=1}^{j} a_i & \text{otherwise;} \end{cases}$$

and for the sequence of functions $\phi_0, \phi_1, \phi_2, \ldots,$

$$\phi_0(n) = n, \ \phi_{\ell+1}(n) = \phi(\phi_\ell(n)),$$

let $\Psi\colon \mathbb{N} \to \Delta = \{0, 1, 2, \ldots, 9\}$ be defined by $\Psi(n) = \phi_\ell(n)$, in which

$$\phi_\ell(n) = \phi_{\ell+1}(n).$$

For example, $\phi(889) = 25$, $\Psi(889) = 7$, since

$$\begin{aligned} \phi_0(889) &= 889, \\ \phi_1(889) &= 25, \\ \phi_2(889) &= 7 \\ &= \phi_3(889). \end{aligned}$$

It then follows that

(3.1) $\quad \Psi(\Psi(10^k/u(0.1)) + k) = 1,$

as Table 1 illustrates.

TABLE 1

k	2	3	4	5	6	7	8	10	11
$\Psi(\underbrace{8 \ldots 89}_{k-1 \text{ times}})$	8	7	6	5	4	3	2	9	8

The result follows from Theorem 1 and 5 of Atanassov, which are, respectively,

(3.2) $\quad \Psi(n + 1) = \Psi(\Psi(n) + 1);$

(3.3) $\quad \Psi(n + 9) = \Psi(n).$

Thus, $10^k/u(1/10) = \underbrace{8 \ldots 89}_{k-1 \text{ times}}$, and so,

$$\Psi(10^k/u(1/10)) = 8(k - 1) + 8 + 1 = 8k + 1,$$

and $\quad \Psi(\Psi(10^k/u(0.1)) + k) = \Psi(9k + 1) = \Psi(9 + 1) = 1$, as required.

4. Other Values of X

The foregoing was for $x = 1/10$. In Table 2, we list the values of $\Psi(f(x))$ for integer values of k and $1/x = X$ from 2 to 10 when $P_j = -1$, $j = 1, 2, \ldots, k$, in the appropriate recurrence relation.

TABLE 2

X/k	2	3	4	5	6	7	8	9	10
2	1	1	1	1	1	1	1	1	1
3	5	5	5	5	5	5	5	5	5
4	2	7	9	8	4	6	5	1	3
5	1	4	1	4	1	4	1	4	1
6	2	2	2	2	2	2	2	2	2
7	5	7	3	2	4	9	8	1	6
8	1	7	1	7	1	7	1	7	1
9	8	8	8	8	8	8	8	8	8
10	8	7	6	5	4	3	2	1	9

To prove these results, we let $x = 1/X$ and so

(4.1) $\quad f(X) = X^k - X^{k-1} - X^{k-2} - \cdots - X^2 - X - 1.$

The calculations which follow are mod 9. Thus, $3^t \equiv 0$, $6^t \equiv 0$, $9^t \equiv 0$ when $t \geq 2$. (Of course, $9^t \equiv 0$ when $t = 1$.)

<u>Case A</u>: $X = 3, 6, 9 = N$,

$f(N) \equiv -N - 1 \pmod 9$ for all k,
$f(3) \equiv -4 \equiv 5$,
$f(6) \equiv -7 \equiv 2$,
$f(9) \equiv -1 \equiv 8$ as in the appropriate rows of Table 2.

<u>Case B</u>: $\quad X = 4, 7, 10 = 3 + 1, 6 + 1, 9 + 1 = N + 1$,

(4.2) $\quad f(N + 1) = (N + 1)^k - (N + 1)^{k-1} - \cdots - (N + 1)^2 - (N + 1) - 1.$

The only terms that interest us, mod 9, in the expansions are the second last and last in each expansion. Then (4.2) becomes

$$Nk - N(k - 1) - N(k - 2) - \cdots - N \cdot 3 - N \cdot 2 - N \cdot 1$$
$$+ 1 - 1 \underbrace{- 1 - 1 - \cdots - 1 - 1 - 1}_{k-2 \text{ times}} - 1$$
$$= Nk - N\sum_{n=1}^{k} n - (k - 2) - 1$$
$$= Nk - \frac{1}{2} N(k - 1)k - (k - 1)$$
$$= Nk\left\{1 - \frac{1}{2}(k - 1)\right\} - (k - 1)$$
$$\equiv Nk^2 - (k - 1) \text{ since } -N \equiv 2N \text{ for } N = 3, 6, 9.$$

Thus, $\quad f(4) = 3k^2 - k + 1$
$\quad f(7) = 6k^2 - k + 1$
$\quad f(10) = -k + 1$ since $9k^2 \equiv 0$.

Substitution of the values $k = 2, 3, \ldots, 10$ gives the tabulated values.

<u>Case C</u>: $\quad X = 2, 5, 8 = 3 - 1, 6 - 1, 9 - 1, = N - 1$,

(4.3) $\quad f(N - 1) = (N - 1)^k - (N - 1)^{k-1} - \cdots - (N - 1)^2 - (N - 1) - 1.$

As in Case B, this becomes

$$Nk(-1)^{k-1} - N(k - 1)(-1)^{k-2} - N(k - 2)(-1)^{k-3} - \cdots - N \cdot 2(-1)^1$$
$$- N \cdot 1(-1)^0 + (-1)^k - (-1)^{k-1} - (-1)^{k-2} - \cdots - 1 + 1 - 1.$$

When k is even, this becomes

$$\underbrace{-Nk - N(k-1)}=\underbrace{N(k-2) - N(k-3)} + \cdots + \underbrace{2N - N} \underbrace{+ 1 + 1}$$
$$\underbrace{- 1 + 1} + \cdots \underbrace{- 1 + 1} - 1 = -2NK + \underbrace{N + N + \cdots + N}_{k/2 \text{ terms}} + 1$$
$$= -2NK + \frac{1}{2}kN + 1$$
$$= -\frac{3}{2}Nk + 1$$
$$\equiv 3Nk + 1 \quad \text{since } -3 \equiv 6$$
$$\equiv 1 \qquad \text{since } 3N \equiv 0,$$

which agrees with the appropriate entries of Table 2.

When k is odd, (4.3) becomes

$$Nk + N(k-1) - N(k-2) + N(k-3) - \cdots - N \cdot 2(-1)^1 - N \cdot 1(-1)^0 - 1$$
$$\underbrace{- 1 + 1}\underbrace{- 1 + 1} - \cdots \underbrace{- 1 + 1} - 1 = Nk + \underbrace{N + N + \cdots + N}_{(k-1)/2 \text{ terms}} - 2$$
$$= Nk + \frac{1}{2}(k-1)N - 2$$
$$= \frac{3}{2}Nk - \frac{1}{2}N - 2$$
$$\equiv -3Nk + 4N - 2 \quad \text{since } 3 \equiv -6,\ -1 \equiv 8$$
$$\equiv 4N - 2 \qquad \text{since } -3N \equiv 0.$$

Thus,

$f(2) \equiv 1,$
$f(5) \equiv 4,$
$f(8) \equiv 7,$ as required.

5. Concluding Comments

Wiliam's staggered sum for Pell numbers [4] can be written as

(5.1) $\quad .0 + .01 + .002 + .0005 + .00012 + .000029 + \cdots = 1/79.$

This is a particular case of Hulbert [5] who also noted a result like (1.3) which can be found in Reichmann [6]. Hulbert stated, without proof, that

(5.2) $\quad \sum_{n=1}^{\infty} 10^{-n} F_n = 1/(9.9 - k)$

for

(5.3) $\quad F_{n+1} = kF_n + F_{n-1}$ with $F_1 - 1$, $F_2 = k$ $(k = 1, 2, \ldots, 8)$.

When $k = 2$, we have the Pell case. We can generalize the Pell sequence by setting $P_1 = 2$, $P_j = -1$, $j = 2, \ldots, k, \ldots$. Then we may extend the work of Section 4 by the addition of a term $-X^{k-1}$ in $f(X)$, for $X = 2, 3, \ldots, 10$.

Hulbert also noted a staggered sum formed from

(5.4) $\quad \sum_{n=1}^{\infty} 10^{-n} \binom{r + n - 1}{r} = 10^{-1}(0.9)^{-r+1}$ $(r = 0, 1, 2, \ldots)$.

This is a particular case of Equation (1.3) of Gould [2], namely

(5.5) $\quad \sum_{r=0}^{\infty} \binom{r + n}{r} x^r = (1 - x)^{-n-1}.$

Curiously, the same issue of the Bulletin where Hulbert's note appeared had in its Puzzle Corner the problem of finding

$$(5.6) \qquad \binom{n}{0} + \binom{n-2}{2} + \binom{n-4}{4} + \cdots$$

the series terminating when the binomial coefficients become improper. This, too, follows from Gould whose Equations (1.74) and (1.75) are, respectively

$$\sum_{k=0}^{[n/2]} \binom{n-k}{k} = (\alpha^{n+1} - \beta^{n+1})/(\alpha - \beta),$$

$$\sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} = \frac{1}{2}((-1)^{[n/3]} + (-1)^{[(n+1)/3]}),$$

where $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, and $[\cdot]$ represents the greatest integer function. It can be seen then that the series (5.6) equals

$$\frac{1}{2} \sum_{k=0}^{[n/2]} (1 + (-1)^k)\binom{n-k}{k}$$
$$= (\alpha^{n+1} - \beta^{n+1})/2(\alpha - \beta) + ((-1)^{[n/3]} + (-1)^{[2(n+1)/3]})/4.$$

It is also of interest to note that the generalized sequences of Section 2 are related to statistical studies of such gambling events as success runs [7] and expected numbers of consecutive heads [3].

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