COMBINATORIAL INTERPRETATIONS OF THE q-ANALOGUES OF L_{2n+1}

A. K. Agarwal*
Indian Institute of Technology, Delhi, New Delhi-110016, India
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1. Introduction

Recently in [1], two different q-analogues of L_{2n+1} were found. Our object here is to interpret these q-analogues as generating functions. As usual, $\begin{bmatrix} n \\ m \end{bmatrix}$ will denote the Gaussian polynomial, which is defined by

$$(1.1)\qquad \begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} (q;q)_n/(q;q)_m(q;q)_{n-m}, & \text{if } 0 \le m \le n, \\ 0, & \text{otherwise} \end{cases}$$

where

$$(a;q)_n = \prod_{i=0}^{\infty} \frac{(1 - aq^i)}{(1 - aq^{n+i})}.$$

We shall also need the following well-known properties of $\begin{bmatrix} n \\ m \end{bmatrix}$:

$$(1.2)\qquad \begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ n - m \end{bmatrix};$$

$$(1.3)\qquad \begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n - 1 \\ m \end{bmatrix} + q^{n-m}\begin{bmatrix} n - 1 \\ m - 1 \end{bmatrix}.$$

In [1], we studied two different q-analogues of L_{2n+1} denoted by $C_n(q)$ and $\overline{C}_n(q)$, respectively. These were defined by

$$(1.4)\qquad C_n(q) = \sum_{j=0}^{n} A_{n,j}(q),$$

where

$$(1.5)\qquad A_{n,j}(q) = \begin{bmatrix} 2n - j \\ j \end{bmatrix} q^{\binom{j}{2}} + (1 + q^j)\begin{bmatrix} 2n - j \\ j - 1 \end{bmatrix} q^{2n-2j+1+\binom{j}{2}}$$

and

$$(1.6)\qquad \overline{C}_n(q) = D_n(q) + D_{n-1}(q).$$

where

$$(1.7)\qquad D_n(q) = \sum_{m=0}^{n} B_{n,m}(q)$$

in which $B_{n,m}(q)$ are defined by

$$(1.8)\qquad B_{n,m}(q) = q^{m^2}\begin{bmatrix} n + m + 1 \\ 2m + 1 \end{bmatrix}.$$

Remark 1: $A_{n,j}(q)$ defined by (1.5) above are $D_{n,j}(q)$ in [1, p. 171] with j replaced by $n - j$. This only reverses the order of summation in (1.4).

Remark 2: Equation (1.8) is (3.6) in [1, p. 172] with m replaced by $n - m$ and (1.2) applied.

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Several combinatorial interpretations of the polynomials $C_n(q)$, $A_{n,m}(q)$, $\overline{C}_n(q)$, $D_n(q)$, and $B_{n,m}(q)$, for $q = 1$, were given in [1]. In this paper, we refine our results for the general value of q, or, in other words, we interpret these polynomials as generating functions. In Section 2, we shall state and prove our main results.

2. The Main Results

In this section, we first state two theorems and three corollaries. The proofs then follow.

Theorem 1: Let $P(m, n, N)$ denote the number of partitions of N into $m - 1$ distinct parts, where the value of each part is less than or equal to $2n - m$, or the number of partitions of N into m distinct parts where each part has a value which is less than or equal to $2n - m + 1$. Then

(2.1) $$A_{n,m}(q) = \sum_{N=0}^{r} P(m, n, N)q^N,$$

where

$$r = 2nm - 3\binom{m}{2}.$$

Example: The coefficient of q^7 in $A_{5,2}(q)$ is 4 (see below); also, $p(2, 5, 7) = 4$, since the relevant partitions are 7, 6 + 1, 5 + 2, and 4 + 3.

$$A_{5,2}(q) = q^{17} + q^{16} + 2q^{15} + 2q^{14} + 3q^{13} + 3q^{12} + 4q^{11} + 4q^{10} + 4q^9 + 4q^8 + 4q^7 + 3q^6 + 3q^5 + 2q^4 + 2q^3 + q^2 + 2$$

Corollary 1:

(2.2) $$C_n(q) = \sum_{N=0}^{s} P(n, N)q^N,$$

where

(2.3) $$P(n, N) = \sum_{m=0}^{n} P(m, n, N)$$

and

$$s = \max\left\{2nm - 3\binom{m}{2}\right\}, \quad 1 \le m \le n.$$

Theorem 2: Let $Q(m, n, N)$ denote the number of partitions of N of the form $\pi = b_1 + b_2 + \cdots + b_t$, such that $m \le t \le 2m + 1$:

$$b_{i-1} - b_i \ge 2 \quad \text{if } 2 \le i \le m$$
$$b_m - b_{m+1} \ge 1$$
$$b_{i-1} \ge b_i \quad \text{if } i > m + 1$$
$$b_1 \le n + m - 1.$$

Then,

(2.4) $$B_{n,m}(q) = \sum_{N=0}^{u} Q(m, n, N)q^N,$$

where

$$u = n^2 + (n - m) - (n - m)^2.$$

Corollary 2:

(2.5) $$D_n(q) = \sum_{N=0}^{n^2} Q(n, N)q^N,$$

where

(2.6) $$Q(n, N) = \sum_{m=0}^{n} Q(m, n, N).$$

Corollary 3:

(2.7) $$\overline{C}_n(q) = \sum_{N=0}^{n^2} R(n, N)q^N,$$

where

(2.8) $$R(n, N) = Q(n, N) + Q(n - 1, N).$$

Proof of Theorem 1: Letting $j = m$ in (1.5), we have

$$A_{n,m}(q) = \left(\begin{bmatrix}2n - m\\ m\end{bmatrix} + \begin{bmatrix}2n - m\\ m - 1\end{bmatrix}q^{2n-2m+1}\right)q^{\binom{m}{2}} + \begin{bmatrix}2n - m\\ m - 1\end{bmatrix}q^{2n-2m+1+m+\binom{m}{2}}$$
$$= \begin{bmatrix}2n - m + 1\\ m\end{bmatrix}q^{\binom{m}{2}} + \left(\begin{bmatrix}2n - m + 1\\ m\end{bmatrix} - \begin{bmatrix}2n - m\\ m\end{bmatrix}\right)q^{\binom{m+1}{2}},$$

where the last step comes by using (1.3) with n replaced by $2n - m + 1$ and noting that

$$m + \binom{m}{2} = \binom{m + 1}{2}.$$

Since $A_{n,m}(q)$ is a polynomial, the degree of $A_{n,m}(q)$ is the degree of

$$\begin{bmatrix}2n - m + 1\\ 2\end{bmatrix}q^{\binom{m+1}{2}}, \text{ which is } 2nm - 3\binom{m}{2}.$$

It is easily seen that

$$\begin{bmatrix}2n - m + 1\\ m\end{bmatrix}q^{\binom{m}{n}}$$

generates partitions into $m - 1$ or m distinct parts, where each part has a value less than or equal to $2n - m$, and

$$\left(\begin{bmatrix}2n - m + 1\\ m\end{bmatrix} - \begin{bmatrix}2n - m\\ m\end{bmatrix}\right)q^{\binom{m+1}{2}}$$

generates partitions into m distinct parts with the largest part equal to $2n - m + 1$. Combining these results, we see that $A_{n,m}(q)$ generates $P(m, n, N)$. The proof of Corollary 1 is now obvious.

Proof of Theorem 2: By the definition of the Gaussian polynomial, it is clear that

$$\begin{bmatrix}n + m + 1\\ 2m + 1\end{bmatrix}$$

generates partitions into at most $2m + 1$ parts where each part has a value less than or equal to $n - m$. Multiplication of $\left[\begin{smallmatrix}n+m+1\\ 2m+1\end{smallmatrix}\right]$ by $q^{m^2} = q^{1+3+\cdots+2m-1}$ means that we are adding $2m - 1$ to the largest part, $2m - 3$ to the next largest part, $2m - 5$ to the next largest part, etc. Since the largest part is less than or equal to $n - m + (2m - 1) = n + m - 1$, there are at least m parts where the minimal difference of the first m parts (with the parts arranged in nonincreasing order) is 2. The m^{th} and the $(m + 1)^{\text{th}}$ parts are distinct. Obviously, the degree of $B_{n,m}(q)$ is

$$m^2 + (2m + 1)(n + m + 1 - 2m - 1) = n^2 + (n - m) - (n - m)^2.$$

This completes the proof of Theorem 2.

Corollaries 2 and 3 are now direct results of Theorem 2.

3. Conclusions

In the literature, we find several combinatorial interpretations of the q-analogues of the Fibonacci numbers. The Catalan numbers and Stirling numbers are other good examples. The most obvious question that arises here is: Do the polynomials $A_{n,m}(q)$, $C_n(q)$, $B_{n,m}(q)$, $D_n(q)$, and $\overline{C}_n(q)$ have combinatorial interpretations other than those presented in this paper? So far, we know one more combinatorial interpretation of the polynomials $D_n(q)$. Before we state it in the form of a theorem, we recall the following definitions from [2].

Definition 1: Let π be a partition. Let γ_{ij} be the node of π in the i^{th} row and j^{th} column of Ferrers' graph of π. We say that γ_{ij} lies on the diagonal δ if $i - j = \delta$.

Definition 2: Let π be a partition whose Ferrers graph is embedded in the fourth quadrant. Each node (i, j) of the fourth quadrant which is not in the Ferrers graph of π is said to possess an anti-hook difference $\rho_i - k_j$ relative to π, where ρ_i is the number of nodes on the i^{th} row of the fourth quadrant to the left of the node (i, j) that are not in the Ferrers graph of π and k_j is the number of nodes in the j^{th} column of the fourth quadrant that lie above node (i, j) and are not in the Ferrers graph of π.

Remark: By the Ferrers graph of a partition, in the above definitions, we mean its graphical representation. If $\pi = a_1 + a_2 + \cdots + a_n$ (with $a_i > a_{i+1}$, $1 < i < n - 1$ is a partition, then the i^{th} row of the graphical representation of this partition contains a_i points (or dots, or nodes). The graphical representation of the partition 5 + 3 + 1 of 9, thus, is:

$$\begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \\ \cdot & & & & \end{array}$$

We now present the other combinatorial interpretation of the polynomials $D_n(q)$ in the following form.

Theorem 3: Let $f(n, k)$ denote the number of partitions of k with the largest part $\le n$ and the number of parts $\le n$, which have all anti-hook differences on the 0 diagonal equal to 0 or 1. Let $g(n, k)$ denote the number of partitions of k with the largest part $\le n + 1$ and the number of parts $\le n - 1$, which have all anti-hook differences on the -2 diagonal equal to 1 or 2. For $k \ge 1$, let $h(n, k) = f(n, k) + g(n, k - 1)$. Then

$$D_n(q) = 1 + \sum_{k=1}^{n^2} h(n, k)q^k.$$

Note: For the proof of Theorem 3, see [2, Th. 2, pts. (1) and (4), p. 11]. We remark here that part (3) of Theorem 2 in [2] was incorrectly stated:

$q^{n^2+n}d_{2n-1}(q^{-1})$ should be replaced by $q^{n^2+n}d_{2n}(q^{-1})$.

References

1. A. K. Agarwal. "Properties of a Recurring Sequence." *The Fibonacci Quarterly* *27.2* (1989):169-75.
2. A. K. Agarwal & G. E. Andrews. "Hook Differences and Lattice Paths." *J. Statist. Plan. Inference* *14* (1986):5-14.

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