ZEROS OF CERTAIN CYCLOTOMY-GENERATED POLYNOMIALS

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1. Introduction

The characteristic equation of the sequence of Fibonacci numbers is

$$x^2 - x - 1 = 0; \tag{1.1}$$

its roots $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$ play an important role in the theory of Fibonacci numbers and other related matters. The Fibonacci numbers have been generalized in various ways. One such generalization and the corresponding characteristic equations were recently studied by Horadam and Shannon [3]:

Define the polynomials $\phi_n(x)$ by $\phi_0 \equiv 0$, and

$$\phi_n(x) = x^{n-1} + x^{n-2} + \cdots + x + 1 \quad (n \geq 1). \tag{1.2}$$

The "cyclotomy-generated polynomial of Fibonacci type" of degree $n^2 + n$ is then defined by

$$p_n(x) = x^{n^2+n} - \phi_{n^2+n} - x^{2n+1}\frac{\phi_{n^2-1}}{\phi_{n+1}} + x^{2n}\frac{\phi_{n^2-n}}{\phi_n}. \tag{1.3}$$

It is easy to see that $p_1(x)$ is the left-hand side of (1.1).

In [3], both real and complex zeros of $p_n(x)$ were studied. However, some of the more interesting properties were given only in the form of conjectures. It is the purpose of this paper to provide proofs of these conjectures, based on some classical results from the geometry of polynomials. Furthermore, it will be shown that the main factor of $p_n(x)$ is irreducible over the rationals for all n, and that the unique positive zeros of $p_n(x)$ are Pisot numbers.

2. Roots of Unity

Horadam and Shannon [3] observed that $n^2 - n$ complex zeros of $p_n(z)$ lie on the unit circle for small n; they conjectured that this is true for all n. The following proves this conjecture.

Proposition 1: $p_n(z)$ has the $n^2 - n$ zeros $z_k = \exp(2\pi ik/n(n+1))$, where $k = 1, 2, \ldots, n^2 + n - 1$, excluding multiples of n and of $n + 1$.

Proof: Note that we may write $\phi_n(x) = (x^n - 1)/(x - 1)$ for $x \neq 1$. With (1.3) we get

$$\begin{aligned}
&(x^{n+1} - 1)(x^n - 1)(x - 1)p_n(x) \\
&= x^{n^2+n}(x^{n+1} - 1)(x^n - 1)(x - 1) - (x^{n^2+n} - 1)(x^{n+1} - 1)(x^n - 1) \\
&\quad - x^{2n+1}(x^{n^2-1} - 1)(x^n - 1)(x - 1) + x^{2n}(x^{n^2-} - 1)(x^{n+1} - 1)(x - 1) \\
&= x^{n^2+3+2} - 3x^{n^2+3+1} + x^{n^2+3} + x^{n^2+2+1} + x^{n^2+2} - x^{n^2+n} \\
&\quad - x^{2n+2} + 3x^{2n+1} - x^{2n} - x^{n+1} - x^n + 1 \\
&= (x^{n^2+n} - 1)(x^{2n+2} - 3x^{2n+1} + x^{2n} + x^{n+1} + x^n - 1),
\end{aligned}$$

and, therefore,

$$p_n(x) = \frac{x^{n^2+n} - 1}{(x^{n+1} - 1)(x^n - 1)(x - 1)}(x^{2n+2} - 3x^{2n+1} + x^{2n} + x^{n+1} + x^n - 1). \tag{2.1}$$

This proves the proposition, since $x^{n^2+n} - 1$ has zeros $z_k = \exp(2\pi ik/n(n + 1))$, where z_n, z_{2n}, ... are cancelled by the zeros of $x^{n+1} - 1$, and z_{n+1}, $z_{2(n+1)}$, ... are cancelled by the zeros of $x^n - 1$.

3. Roots within the Unit Circle

It is clear from (2.1) that the remaining zeros of $p_n(z)$ are those of

$$f_n(z) := z^{2n+2} - 3z^{2n+1} + z^{2n} + z^{n+1} + z^n - 1. \tag{3.1}$$

First, we note that $f_n(z)$ has a double zero at $z = 1$, since $z^{n+1} - 1$ and $z^n - 1$ have simple zeros at $z = 1$, while $p_n(1) = 1 - n^2 - n \neq 0$, by (1.3). Hence, we may consider

$$\begin{aligned} r_n(z) &:= f_n(z)/(z - 1)^2 \\ &= z^{2n} - z^{2n-1} - 2z^{2n-2} - \cdots - nz^n - nz^{n-1} - (n - 1)z^{n-2} \\ &\qquad - \cdots - 2z - 1 \end{aligned} \tag{3.2}$$

(see also [3, p. 91]). We note that we can write

$$r_n(z) = z^{2n} - \frac{(1 - z^n)(1 - z^{n+1})}{(1 - z)^2}. \tag{3.3}$$

The following three propositions show that all but one of the zeros of $r_n(z)$ lie in a narrow annular region just inside the unit circle, and that the arguments of all $2n$ zeros are quite evenly distributed.

Proposition 2: For all $n \geq 1$, the zeros of $r_n(z)$ lie outside the circle

$$|z| = (1/3)^{1/n}.$$

Proof: We apply Rouché's Theorem (see, e.g., [4, p. 2]). Departing from (3.3), we let

$$P(z) := z^{2n} \quad \text{and} \quad Q(z) := -(1 - z^n)(1 - z^{n+1})/(1 - z)^2.$$

Set $t := |z|$. Now, for $t < 1$,

$$|Q(z)| \geq \frac{(1 - t^n)(1 - t^{n+1})}{(1 + t)^2} = \frac{1 - t^n - t^{n+1} + t^{2n+1}}{(1 + t)^2},$$

while

$$|P(z)| = t^{2n}.$$

Hence, we have $|Q(z)| > |P(z)|$ when

$$\frac{1 - t^n - t^{n+1} + t^{2n+1}}{(1 + t)^2} > t^{2n},$$

which is equivalent to

$$t^n(1 + t + t^n + t^{n+1} + t^{n+2}) < 1;$$

this holds when

$$t^n(2 + 3t^n) \leq 1$$

(since $t < 1$). But this last inequality is satisfied for $t^n = 1/3$. Hence, by Rouché's Theorem, $r_n(z) = P(z) + Q(z)$ has the same number of zeros within the circle $|z| = (1/3)^{1/n}$ as does $Q(z)$, namely, none at all, since all the zeros of $Q(z)$ have modulus 1. Also, the above inequalities show that there can be no zero on this circle. The proof is now complete.

Proposition 3: For $n \geq 1$, $r_n(z)$ has $2n - 1$ zeros within the unit circle.

Proof: It is easy to verify the following factorization. For any a,

$$(3.4)\qquad (a + 1)az^{2n} - z^{2n-1} - 2z^{2n-2} - \cdots - nz^n - nz^{n-1} - (n - 1)z^{n-2} - \cdots - 2z - 1$$

$$= [(a + 1)z^n + z^{n-1} + \cdots + z + 1][az^n - z^{n-1} - \cdots - z - 1].$$

In particular, if we set $a = (\sqrt{5} - 1)/2$, then $(a + 1)a = 1$, and with (3.1) we get

$$(3.5)\qquad r_n(z) = g_n(z)h_n(z),$$

where

$$g_n(z) = \frac{\sqrt{5} - 1}{2}z^n - z^{n-1} - \cdots - z - 1,$$

and

$$h_n(z) = \frac{\sqrt{5} + 1}{2}z^n + z^{n-1} + \cdots + z + 1.$$

The Kakeya-Eneström Theorem (see, e.g., [4, p. 136] or [7, Prob. III.22]) now shows immediately that all n zeros of $h_n(z)$ lie within the unit circle. To deal with $g_n(z)$, we consider

$$(3.6)\qquad (z - 1)g_n(z) = \frac{\sqrt{5} - 1}{2}z^{n+1} - \frac{\sqrt{5} + 1}{2}z^n + 1.$$

By Pellet's Theorem (see, e.g., [4, p. 128]), n zeros of $(z - 1)g_n(z)$ lie on or within the unit circle. But $z = 1$ is the only zero on the unit cricle, since the difference of the complex vectors $((\sqrt{5} - 1)/2)z^{n+1}$ and $((\sqrt{5} - 1)/2z^n$ has length one only if they are collinear; (3.6) then implies $z = 1$. Hence, $g_n(z)$ has $n - 1$ zeros within the unit circle. (We remark that this fact also follows directly from Theorem 2.1 in [2].) The proof is now complete, with (3.5).

It was remarked in [3] that the complex zeros of $p_n(z)$ not located on the unit circle appear to lie close to the "missing" roots of unity (see Proposition 1 above). With regard to this, we have the following result.

Proposition 4: For $n \geq 1$, $r_n(z)$ has at least one zero in each sector

$$\left|\arg z - \frac{k}{n}\pi\right| \leq \frac{\pi}{n + 1},\ k = 0, 1, \ldots, 2n - 1.$$

Proof: We use the factorization (3.5). In analogy to (3.6), we have

$$(3.7)\qquad (z - 1)h_n(z) = \frac{\sqrt{5} + 1}{2}z^{n+1} - \frac{\sqrt{5} - 1}{2}z^n - 1.$$

Equation (3.7) can be brought into the form $az^{n+1} + z^n + 1$ by replacing z by $((\sqrt{5} + 1)/2)^{1/n}z$. The result of [4, p. 165, Ex. 3] implies that $(z - 1)h_n(z)$ has at least one zero in each of the sectors

$$(3.8)\qquad \left|\arg z - \frac{2k + 1}{n}\pi\right| \leq \frac{\pi}{n + 1},\ k = 0, 1, \ldots, n - 1.$$

The trivial zero $z = 1$ (i.e., $\arg z = 0$) is not contained in any of these sectors; hence, exactly one zero of $h_n(z)$ lies in each of the sectors (3.8).

To deal with the factor $g_n(z)$, we consider (3.6) and replace z by

$$(e^{i\pi}(\sqrt{5} - 1)/2)^{1/n}z.$$

This brings the right-hand side of (3.6) into the form $a'z^{n+1} + z^n + 1$ for some complex a'. We now apply a well-known result on the angular distribution of the zeros of certain trinomials (see [4, p. 165, Ex. 3]) and "rotate" the resulting sectors by an angle of π/n. This shows that $(z - 1)g_n(z)$ has at least one zero in each of the sectors

$$\left|\arg z - \frac{2k}{n}\pi\right| \le \frac{\pi}{n+1}, \quad k = 0, 1, \ldots, n - 1. \tag{3.9}$$

But the sector belonging to $k = 0$ contains two zeros, namely, $z = 1$ and the unique positive zero of $g_n(z)$ (by Descartes's Rule of Signs; see, e.g., [6, p. 45]). Hence, each sector (3.9) contains exactly one zero of $g_n(z)$. This proves Proposition 4. [We have actually proved a slightly stronger statement; but the sectors (3.8) and (3.9) are overlapping.]

Remark: As we just saw, the trinomials on the right-hand sides of (3.6) and (3.7) can be brought into the form $f(z) = az^{n+1} + z^n + 1$. One could also consider the inverted polynomial

$$f^*(z) = z^{n+1}\overline{f}(1/z) = z^{n+1} + z + \overline{a},$$

the zeros z_j^* of which are the inverses of the zeros z_j of $f(z)$ relative to the unit circle (i.e., $z_j^* = 1/\overline{z}_j$; see [4, p. 194]). In this regard, we mention that the trinomials $z^{n+1} - (n + 1)z + n = 0$ were studied in [5]; very exact bounds on the arguments and the moduli of the zeros of these trinomials were obtained. Probably the methods in [5] could be used to obtain similar results for the trinomials in (3.6) and (3.7).

4. Real Roots

Horadam and Shannon [3] showed that $p_n(z)$ has exactly one positive zero x_{1n} with

$$\lim_{n\to\infty} x_{1n} = \frac{\sqrt{5}+3}{2} \left[= \left(\frac{\sqrt{5}+1}{2}\right)^2\right]; \tag{4.1}$$

this is the one zero not covered by Proposition 3. They also conjectured that $p_n(z)$ has exactly one negative zero x_{2n} with $-1 < x_{2n} < 0$. They proved this conjecture under the condition that the factorization (2.1) above is true; hence, the existence of this negative zero is established. It was also conjectured in [3] that

$$\lim_{n\to\infty} x_{2n} = -1; \tag{4.2}$$

this follows immediately from Propositions 2 and 3. Our aim in this section is to give quantitative versions of (4.1) and (4.2).

Let $G_n(z)$ and $H_n(z)$ denote the trinomials on the right-hand sides of (3.6) and (3.7), respectively. By Descartes's Rule of Signs, $G_n(z)$ has two positive zeros ($z = 1$ and $z = x_{1n}$), while $H_n(z)$ has only one positive zero ($z = 1$). As to the negative zeros, we consider $G_n(-z)$ and $H_n(-z)$. The signs of the coefficients of $G_n(-z)$ are $(-1)^{n+1}$, $(-1)^{n+1}$, 1; that is, there is one sign change when n is even and none when n is odd. Hence, $G_n(z)$ has a negative zero (namely, x_{2n}) only when n is even. The signs of the coefficients of $H_n(-z)$ are $(-1)^{n+1}$, $(-1)^{n+1}$, -1; this implies that $H_n(z)$ has a negative zero (namely, x_{2n}) only when n is odd.

The following results give estimates on the location of these zeros.

Proposition 5: Let $\alpha := (\sqrt{5} + 1)/2$. Then, for all $n \ge 1$,

$$\alpha^2(1 - \alpha^{-2n}) \le x_{1n} < \alpha^2(1 - \alpha^{-2n-1}),$$

with equality only for $n = 1$. Furthermore, we have, asymptotically,

$$x_{1n} \sim \alpha^2(1 - \alpha^{-2n-1}) \text{ as } n \to \infty.$$

Proof: It suffices to find two points at which $G_n(z)$ has opposite signs. It is easy to see that, for any ε, we have

$$G_n(\alpha^2 - \varepsilon) = -\varepsilon \frac{\sqrt{5} - 1}{2}(\alpha^2 - \varepsilon)^n + 1,$$

and therefore, for arbitrary numbers a,

$$G_n(\alpha^2 - a\alpha^{-2n}) = -a \frac{\sqrt{5} - 1}{2}(1 - a\alpha^{-2n-2})^n + 1. \tag{4.3}$$

First, we let $a = \alpha$. Since $(1 - \alpha^{-2n-1})^n < 1$ for all n, we get

$$G_n(\alpha^2 - \alpha^{1-2n}) > 0 \text{ for } n \geq 1. \tag{4.4}$$

In the other direction, we set $a = \alpha^2$. It is easy to see (using calculus) that $(1 - \alpha^{-2n})^n$ is an increasing sequence for $n \geq 1$. Thus, we get, with (4.3),

$$G_n(\alpha^2 - \alpha^{2-2n}) \leq G_1(\alpha^2 - 1) = 0,$$

with equality only for $n = 1$. This, together with (4.4), proves the first statement of the proposition. The asymptotic expression follows from the fact that, for any real a, we have $(1 - a\alpha^{-2n-2})^n \to 1$ as $n \to \infty$.

Proposition 6: For all $n \geq 2$, we have

$$-1 + \frac{1}{2n} < x_{2n} < -1 + \frac{\log 5}{2n}, \tag{4.5}$$

and we have, asymptotically,

$$x_{2n} \sim -1 + \frac{\log 5}{2n} \text{ as } n \to \infty.$$

Proof: First, let n be even. Then, for any a, we have

$$G_n\left(-1 + \frac{a}{n}\right) = -\left(1 - \frac{a}{n}\right)^n \left[\sqrt{5} - \frac{a}{n}\,\frac{\sqrt{5} - 1}{2}\right] + 1. \tag{4.6}$$

We note that $(1 - a/n)^n$ is an increasing sequence for $n \geq 2$, at least when $a = 1/2$. Hence, for all $n \geq 2$,

$$G_n\left(-1 + \frac{1/2}{n}\right) \leq G_2\left(-1 + \frac{1}{4}\right) = -\left(1 - \frac{1}{4}\right)^2 \left[\sqrt{5} - \frac{\sqrt{5} - 1}{8}\right] + 1 < 0.$$

In the other direction, we use the fact that $(1 - (\log 5)/2n)^n < 1/\sqrt{5}$ for all n. Hence, with (4.6),

$$G_n\left(-1 + \frac{\log 5}{2n}\right) > -\frac{1}{\sqrt{5}}\sqrt{5} + 1 = 0.$$

This proves (4.5) for even n. If n is odd, we have, for arbitrary a,

$$H_n\left(-1 + \frac{a}{n}\right) = \left(1 - \frac{a}{n}\right)^n \left[\sqrt{5} - \frac{a}{n}\,\frac{\sqrt{5} + 1}{2}\right] - 1. \tag{4.7}$$

We find that, for $n \geq 3$,

$$H_n\left(-1 + \frac{1/2}{n}\right) \geq H_3\left(-1 + \frac{1}{6}\right) = \left(1 - \frac{1}{6}\right)^3 \left[\sqrt{5} - \frac{\sqrt{5} + 1}{12}\right] - 1 > 0,$$

while, again with (4.7),

$$H_n\left(-1 + \frac{\log 5}{2n}\right) < \frac{1}{\sqrt{5}}\sqrt{5} - 1 = 0.$$

This completes the proof of (4.5). The asymptotic expression follows from (4.6) and (4.7), and from the fact that $(1 - a/n)^n \to e^{-a}$ as $n \to \infty$.

Remarks: *(1)* The zero $x_{21} = -1/\alpha \simeq -0.61803$; it does not satisfy (4.5).

(2) As an illustration for Propositions 5 and 6, see Table 3 in [3]. The two results also explain the observation in [3] that "the negative root approaches its lower bound more slowly than the positive root approaches its upper bound."

5. Some Algebra

In the theory of uniform distribution modulo 1, sequences of the type ω^n supply important special cases (see, e.g., [8, p. 2]). For instance, it is known that ω^n is uniformly distributed modulo 1 for almost all (in the Lebesgue sense) numbers $\omega > 1$, but very little is known for particular values of ω. On the other hand, it is of interest to study "bad" examples of ω, namely, those for which the sequence ω^n is very "unevenly" distributed modulo 1.

One such example is $\omega = \alpha = (1 + \sqrt{5})/2$; its conjugate is $\beta = (1 - \sqrt{5})/2$. Now $\alpha^n + \beta^n$ are the Lucas numbers 2, 1, 3, 4, 7, ... and thus are rational integers, so that

$$\alpha^n + \beta^n \equiv 0 \pmod 1.$$

But $|\beta| < 1$, and so $\beta^n \to 0$ as $n \to \infty$, which implies that $\alpha^n \to 0 \pmod 1$.

Hence, α^n (modulo 1) has a single accumulation point. α shares this property with a wider class of algebraic numbers (see [8] or [1]).

Definition: A *Pisot number* is an algebraic integer $\theta > 1$ such that all of its conjugates have moduli strictly less than 1.

Theorem (Salem [8]): If θ is a Pisot number, then $\theta \;\to 0 \pmod 1$ as $n \to \infty$.

The proof of this theorem is similar to the above discussion on the properties of α^n.

It is our aim now to show that the unique positive zeros x_{1n} of the polynomials $p_n(x)$ are Pisot numbers. First, we need the following result,

Proposition 7: The polynomials $r_n(x)$ are irreducible over the rationals.

We have seen in the previous sections that $r_n(z)$ has $2n - 1$ zeros satisfying $|z| < 1$ and one zero satisfying $|z| > 1$. Also, $r_n(z)$ is a monic polynomial with rational integer coefficients. If $r_n(z)$ were reducible over the rationals, then, by Gauss's Lemma, $r_n(z) = G(z)H(z)$ for suitable monic polynomials $G(z)$, $H(z)$ of positive degrees with rational integer coefficients. One of these polynomials, say $G(z)$, must have all its zeros of modulus strictly less than one. Hence, the constant term of $G(z)$ (the product of all its zeros) has modulus $|G(0)| < 1$. But this contradicts the fact that the constant term of $G(z)$ is a nonzero integer.

Remark: The proof of Proposition 7 is taken from [9] where, by the way, a trinomial similar to (3.6) and (3.7) is considered. See also the remark on page 12 in [1].

Proposition 8: The unique positive zeros x_{1n} of $r_n(z)$ are Pisot numbers for all $n \geq 1$.

Proof: This follows from Proposition 7 and the results on the zeros of $r_n(z)$ in the previous sections.

We close with a factorization involving Fibonacci numbers. Equation (3.4) shows that the left-hand side of (3.4) splits into two factors of equal degree if a is rational. On the other hand, Proposition 7 shows that this polynomial is irreducible over $\mathbb{Q}$ for $a = (\sqrt{5} - 1)/2$. These remarks suggest that we set

$a = F_k/F_{k+1}$ (where F_k is the k^{th} Fibonacci number), as it is well known that

$$F_k/F_{k+1} \to (\sqrt{5} - 1)/2 \text{ for } k \to \infty;$$

these fractions are actually the best rational approximations to $(\sqrt{5} - 1)/2$. If we take into account

$$a + 1 = F_k/F_{k+1} + 1 = F_{k+2}/F_{k+1}$$

and

$$(a + 1)a = F_{k+2}F_k/F_{k+1}^2 = (F_{k+1}^2 - 1)/F_{k+1}^2,$$

we obtain the factorization

$$\begin{aligned}(1 - F_{k+1}^{-2})z^{2n} - z^{2n-1} - 2z^{2n-2} - \cdots - nz^n - nz^{n-1} - (n - 1)z^{n-2}\\ - \cdots - 2z - 1\\ = [(F_{k+2}/F_{k+1})z^n + z^{n-1} + \cdots + z + 1][(F_k/F_{k+1})z^n - z^{n-1}\\ - \cdots - z - 1].\end{aligned}$$

We note that the left-hand side of this factorization converges quite rapidly to $r_n(z)$ as $k \to \infty$, uniformly on compact subsets of $\mathbb{C}$.

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