FIBONACCI'S MATHEMATICAL LETTER TO MASTER THEODORUS

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(Submitted March 1989)

1. Introduction

Sometime about 1225 A.D., Fibonacci—or Leonardo of Pisa, as he was known until relatively recent times—wrote an interesting, undated mathematical letter to Master Theodorus, philosopher at the court of the Holy Roman Emperor, Frederick II. The full title of this communication, written in medieval Tuscan Latin, is: *Epistola suprascripti Leonardi ad Magistrum Theodorum phylosophum domini Imperatoris*.

Our knowledge of this epistle comes from the nineteenth-century publication of Fibonacci's manuscripts by Boncompagni [1], which is the first printed record of his works. Boncompagni's printing of the *Epistola* [1, pp. 247-52] was prepared from a manuscript in the Biblioteca Ambrosiana di Milano. It has never been translated into English, though in 1919 McClenon [8] indicated his intention to do so.

Fibonacci's mathematical writings consist of five works (others having been lost). These are: (1) *Liber abbaci* (1202, revised 1228); (2) *Practica geometriae* (1220); (3) *Flos* (1225); (4) *Liber quadratorum* (1225), his greatest opus; and (5) the letter to Master Theodorus, the shortest of his extant writings. This useful letter has been somewhat neglected by historians of mathematics, a tendency I would like to see reversed.

To understand Fibonacci's outstanding contributions to knowledge, it is necessary to know something of the age in which he lived and of the mathematics that preceded him. Indeed, a study of his writings reminds one of the history of pre-medieval mathematics in microcosm. In an age of great commercial change and expansion, as well as political and religious struggle, he traveled widely throughout the Mediterranean area in pursuit of his business and mathematical interests. His writings reflect many sources of influence, principally the Greeks in geometry and number theory and the Arabs in algebraic techniques, while some of his problems reveal oriental influences emanating from China and India. Babylonian and Egyptian ideas are apparent in his calculations. For further information on Fibonacci's life and times one may consult, for example, Gies & Gies [3], Grimm [4], Herlihy [5], and Horadam [6].

In popular estimation, Fibonacci is best known for his introduction to Europe of the Hindu-Arabic numerals and, of course, for the set of integers associated (in the late nineteenth century) with his name. However, these popular images of Fibonacci obscure the consummate mastery he demonstrated in a wide range of mathematics.

2. The Letter to Master Theodorus

The mathematical contents of the *Epistola* are rather more speculative and recreational than is the material of his two major, earlier works which have an emphasis on practical arithmetic and geometry. After some general introductory remarks directed to Master Theodorus, Fibonacci proceeds to pose, and solve, a variety of problems.

(a) Problems of Buying Birds

In the first section of this document, Fibonacci's main subject is the "Problem of the 100 birds," a type of problem of oriental origin which he had previously discussed in *Liber abbaci*. Here, however, he develops a general method for solving indeterminate problems.

Fibonacci begins by discussing variations of the problem of buying a given number of birds (sparrows, turtledoves, and pigeons—let us label them x, y, and z—costing 1/3, 1/2, and 2 *denarii* each, respectively) with a given number of *denarii*, a *denarius* being a coin unit of currency. Details of the cases may be tabulated thus:

Denarii	Birds	Solution(s): x, y, z
30	30	9 10 11
29	29	3 16 10; 12 6 11
15	15	
16	15	3 6 6

Regarding the third case, which is insoluble in integers (mathematically, we obtain $4\frac{1}{2}$, 5, $5\frac{1}{2}$), Fibonacci remarks: "...*hoc esse non posse sine fractione avium demonstrabo*."

Next, Fibonacci varies the cost per bird when buying 30 birds of 3 kinds with 30 *denarii*. A bird of each variety now costs 1/3, 2, and 3 *denarii* respectively. He finds the unique solution to be 21, 4, 5 for the numbers of each kind of bird.

Finally, Fibonacci deals with the purchase with 24 *denarii* of birds of 4 kinds (sparrows, turtledoves, pigeons, and partridges) at a specified cost per bird, leading to the equations

$$x + y + z + t = 24,$$

$$\frac{1}{5}x + \frac{1}{3}y + 2z + 3t = 24,$$

for which the two solutions are 10, 6, 4, 4, and 5, 12, 2, 5, for x, y, z, t, respectively.

Admittedly, these problems become somewhat tedious because of their repetitive nature, but an insight into Fibonacci's mind is revealing. Remember that he had no algebraic symbolism to guide him. While his techniques, supplemented by tabulated information in the margin, are fairly standard for us in these problems, they might not have been easy for him.

(b) A Geometrical Problem

Following these algebraic problems, Fibonacci [1 (Vol. 2), p. 249] then proposes the geometrical construction of an equilateral pentagon in a given isosceles ("equicrural," i.e., equal legs) triangle. [Observe that an equilateral pentagon is only regular if it is also equiangular (108°).]

This problem in Euclidean geometry will be highlighted, for historical reasons, and for variety. Fibonacci states the problem in these words: "*De compositione pentagonj equilateri in triangulum equicrurium datum*."

Our Figure 1 reproduces Fibonacci's diagram and notation. In it, Fibonacci takes $ab = ac = 10$, $bc = 12$, and draws di, ah, gl perpendicular to the base bc. The equilateral pentagon is $adefg$. Taking the side ad of the pentagon as the unknown *res* ("thing")—our x—to be determined, and using similar triangles, Fibonacci applies Pythagoras' Theorem to the triangle die to obtain

$$\left(8 - \frac{4}{5}x\right)^2 + \left(\frac{1}{10}x\right)^2 = x^2,$$

whence,

$$x^2 + 36\tfrac{4}{7}x = 182\tfrac{6}{7}$$

("*et sic reducta est questio ad unam ex regulis algebre*," he writes).

He obtains the approximate value

$$x = 4^0\,27^{i}\,24^{ii}\,40^{iii}\,50^{iv}$$

in Babylonian sexagesimal notation.

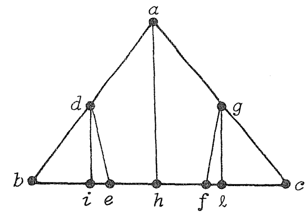

FIGURE 1

To achieve his solution, Fibonacci, with the visual aid of a geometrical diagram involving a square and rectangles, completes the square in the quadratic, i.e.,

$$\left(x + 18\tfrac{2}{7}\right)^2 = 517\tfrac{11}{49}\ \left[= 182\tfrac{6}{7} + \left(18\tfrac{2}{7}\right)^2\right],$$

then subtracts $18\frac{2}{7}$ from the square root of $517\frac{11}{49}$ (which he gives in sexagesimal notation as approximately "*22 et minuta 44 et secunda 23 et tertia 13 et quarta 7*," i.e., $22^0\,44^{i}\,23^{ii}\,13^{iii}\,7^{iv}$).

According to my computations using a calculator, Fibonacci's sexagesimal approximation agrees to six decimal places with my approximation (4.456855).

Fibonacci's problem is wrongly stated by some writers, for example, Van der Waerden [10, p. 40] and Vogel [11, p. 610], both of whom say: "A regular pentagon is (to be) inscribed in an equilateral triangle." (How can angles of 108° and 60° be equated?) It is all the more surprising to have Vogel immediately afterward praise Fibonacci's treatment as "a model for the early application of algebra in geometry" (a statement with which one cannot, of course, disagree). Perhaps the error is due to a mistranslation.

Loria [7, p. 231], who does give a proper account of the problem, states however that x is the length *bd* (which may be a misprint). But Fibonacci, after saying that he is taking each side of the pentagon to be *res*, continues "...*et auferam ad ex ab, scilicet rem de 10, remanebit db 10 minus re*" (i.e., $ab - ad = db = 10 - x$), i.e., $ad = x$.

Cantor [2] gives a correct interpretation and analysis of the problem. (Additionally, he extends Fibonacci's problem by finding the value of x in terms of equal sides of length a and base-length b for the general isosceles triangle.)

(c) Problems on the Distribution of Money

After this excursion into Euclidean geometry, Fibonacci reverts to money problems, in particular the distribution of money among five men—let us designate them by x, y, z, u, v—according to certain prescribed conditions.

In effect, we are required to solve the five equations

$$x + \frac{1}{2}y = 12,\ y + \frac{1}{3}z = 15,\ z + \frac{1}{4}u = 18,\ u + \frac{1}{5}v = 20,\ v + \frac{1}{6}x = 23.$$

To assist his explanation, Fibonacci arranges some of the information in tabular form. The answers are:

$$x = 6\tfrac{612}{721},\ y = 10\tfrac{218}{721},\ z = 14\tfrac{67}{721},\ u = 15\tfrac{453}{721},\ v = 21\tfrac{619}{721}.$$

However, Fibonacci presents his solutions in the Arabic form, i.e., the fractions precede the integer. For example, he gives v as $\frac{3}{7}\ \frac{88}{103}\ 21$, where the fractional part is to be interpreted as

$$\frac{3 + 88 \times 7}{7 \times 103}.$$

(I cannot reconcile my correct answer for x with the printed version of Fibonacci's answer which is not easy to decipher in my enlarged photocopy of the microfilmed text.) Fibonacci's argument in his solution indicates that he is thinking of the calculations for each man being performed in columns. Apart from this technique, his method of solution is the usual mechanical one of clearing the given equations of fractions and then adding or subtracting successive pairs of equations as appropriate.

The letter concludes with a variation of this problem. Fibonacci now requires the reader to solve the system of five equations:

$$x + \frac{1}{2}(y + z + u + v) = 12,$$

$$y + \frac{1}{3}(z + u + v + x) = 15,$$

$$z + \frac{1}{4}(u + v + x + y) = 18,$$

$$u + \frac{1}{5}(v + x + y + z) = 20,$$

$$v + \frac{1}{6}(x + y + z + u) = 23.$$

He does not tell us how he resolves the problem but finishes his correspondence with the comment: "*tunc questio esset insolubilis, nisi concederetur, primus habere debitum; quod debitum esset* $\frac{97}{197}$ 13."

His correct, unique solution is, in our notation,

$$x = -13\frac{97}{197},\ y = 3\frac{297}{394},\ z = 11\frac{99}{197},\ u = 15\frac{247}{394},\ v = 20\frac{20}{197}.$$

Much computational skill must have been required to achieve this solution. What is also important is the fact that Fibonacci was willing to acknowledge a negative number as a solution, this negative number being conceived in commercial terms as a debt. He did not, of course, use the minus sign which was introduced via mercantile arithmetic in Germany nearly three centuries later (also to represent a debt).

3. Concluding Remarks

While Fibonacci's letter to Master Theodorus does not reveal the true magnitude of his genius, it does exhibit some of his originality, versatility, and wide-ranging expertise, as well as some of his powerful methods.

He was, indeed, the *primum mobile* in pioneering the rejuvenation of mathematics in Christian Europe. He absorbed, and independently extended, the knowledge of his precursors, demonstrating a particular agility with computations and manipulations with indeterminate equations of the first and second degrees. In his geometrical expositions, he displayed a complete mastery of the content and rigor of Euclid's works and, moreover, he applied to problems of geometry the new techniques of algebra.

Unquestionably he was, as competent critics agree, the greatest creator and exponent of number theory for over a millennium between the time of Diophantus and that of Fermat.

To measure one's own mathematical ability against that of Fibonacci (born about 1175, died about 1240, while Pisa was still a prosperous maritime republic), the reader is invited to attempt some of the problems occurring in Fibonacci's writings, especially his *Liber quadratorum* (see Sigler [9]), e.g., the last problem in that book—proposed by Master Theodorus—namely, to solve the equations

$$x + y + z + x^2 = u^2,$$
$$x + y + z + x^2 + y^2 = v^2,$$
$$x + y + z + x^2 + y^2 + z^2 = w^2.$$

I conclude this short treatment of the *Epistola* with a chastening quote [4] from Fibonacci's best-known work, *Liber abbaci*, which expresses a sentiment reiterated in the Prologue of *Liber quadratorum* [9]:

> If I have perchance omitted anything more or less proper or necessary, I beg indulgence, since there is no one who is blameless and utterly provident in all things.

References

1. B. Boncompagni. *Scritti di Leonardo Pisano*. 2 vols. Rome: 1857 (Vol. I); 1862 (Vol. II).
2. M. Cantor. *Vorlesungen über Geschichte der Mathematik. Zweiter Band*. Leipzig: Teubner, 1913.
3. J. Gies & F. Gies. *Leonard of Pisa and the New Mathematics of the Middle Ages*. New York: Crowell, 1969.
4. R. E. Grimm. "The Autobiography of Leonardo Pisano." *Fibonacci Quarterly 11.1* (1973):99-104.
5. D. Herlihy. *Pisa in the Early Renaissance: A Study of Urban Growth*. New Haven: Yale University Press, 1958.
6. A. F. Horadam. "Eight Hundred Years Young." *The Australian Mathematics Teacher 31.4* (1975):123-34.
7. G. Loria. *Storia della Matematiche*. Milan: Editore Ulrico Hoepli, 1950.
8. R. B. McClenon. "Leonardo of Pisa and His *Liber Quadratorum*." *Amer. Math. Monthly 26.1* (1919):1-8.
9. L. E. Sigler. "The Book of Squares" (trans. of Fibonacci's *Liber quadratorum*). New York: Academic Press, 1987.
10. B. L. Van der Waerden. *A History of Algebra from al-Khowārismī to Emmy Noether*. New York: Springer-Verlag, 1985.
11. K. Vogel. "Fibonacci, Leonardo, or Leonardo of Pisa." *Dictionary of Scientific Biography*. Vol. 14. New York: Scribner, 1971, pp. 604-13.
12. A. P. Youschkevitch. *Geschichte der Mathematik im Mittelalter*. Leipzig: Teubner, 1964 (trans. from the Russian).

Addendum: A translation of Fibonacci's letter with commentary (both in Italian) appears in E. Picutti, "Il 'Flos' di Leonardo Pisano," *Physis 25* (1983): 293-387.

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