CHOLESKY ALGORITHM MATRICES OF FIBONACCI TYPE AND PROPERTIES OF GENERALIZED SEQUENCES*

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1. Introduction

Many properties of the generalized Fibonacci numbers U_n and the generalized Lucas numbers V_n (e.g., see [3]-[5], [8]-[10], [12], [14], [15]) have been obtained by altering their recurrence relation and/or the initial conditions. Here we offer a somewhat new matrix approach for developing properties of this nature.

The aim of this paper is to use the 2-by-2 matrix $\mathbf{M}_k$ determined by the Cholesky LR decomposition algorithm to establish a large number of identities involving U_n and V_n. Some of these identities, most of which we believe to be new, extend the results obtained in [6] and elsewhere.

Particular examples of the use of the matrix $\mathbf{M}_k$, including summation of some finite series involving U_n and V_n, are exhibited. A method for evaluating some infinite series is then presented which is based on the use of a closed form expression for certain functions of the matrix $x\mathbf{M}_k^n$.

2. Generalities

In this section some definitions are given and some results are established which will be used throughout the paper.

2.1. The Numbers U_n and the Matrix M

Letting m be a natural number, we define (see [4]) the integers $U_n(m)$ (or more simply U_n if there is no fear of confusion) by the second-order recurrence relation

(2.1) $\quad U_{n+2} = mU_{n+1} + U_n; \quad U_0 = 0,\ U_1 = 1 \quad \forall\, m.$

The first few numbers of the sequence $\{U_n\}$ are:

U_0	U_1	U_2	U_3	U_4	U_5	U_6	...
0	1	m	m^2+1	m^3+2m	m^4+3m^2+1	m^5+4m^3+3m	... .

We recall [4] that the numbers U_n can be expressed in the closed form (Binet form)

(2.2) $\quad U_n = (\alpha_m^n - \beta_m^n)/\Delta_m,$

where

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$$(2.3)\qquad \begin{cases} \Delta_m = \sqrt{m^2+4} \\ \alpha_m = (m+\Delta_m)/2 \\ \beta_m = (m-\Delta_m)/2. \end{cases}$$

From (2.3) it can be noted that

$$(2.4)\qquad \begin{cases} \alpha_m\beta_m = -1 \\ \alpha_m + \beta_m = m \\ \alpha_m - \beta_m = \Delta_m. \end{cases}$$

We also recall [4] that

$$(2.5)\qquad U_n = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} m^{n-2j-1},$$

where $[\cdot]$ denotes the greatest integer function. Moreover, as we sometimes require negative-valued subscripts, from (2.2) and (2.4) we deduce that

$$(2.6)\qquad U_{-n} = (-1)^{n+1}U_n.$$

From (2.1) it can be noted that the numbers $U_n(1)$ are the Fibonacci numbers F_n and the numbers $U_n(2)$ are the Pell numbers P_n.

Analogously, the numbers $V_n(m)$ (or more simply V_n) can be defined [4] as

$$(2.7)\qquad V_n = \alpha_m^n - \beta_m^n = U_{n-1} + U_{n+1}.$$

The first few numbers of the sequence $\{V_n\}$ are:

V_0	V_1	V_2	V_3	V_4	V_5	V_6	...
2	m	m^2+2	m^3+3m	m^4+4m^2+2	m^5+5m^3+5m	$m^6+6m^4+9m^2+2$	... .

These numbers satisfy the recurrence relation

$$(2.8)\qquad V_{n+2} = mV_{n+1} + V_n;\quad V_0 = 2,\ V_1 = m \quad \forall m.$$

From (2.7) and (2.4) we have

$$(2.9)\qquad V_{-n} = (-1)^n V_n,$$

and it is apparent that the numbers $V_n(1)$ are the Lucas numbers L_n while the numbers $V_n(2)$ are the Pell-Lucas numbers Q_n [11].

Definitions (2.1) and (2.8) can be extended to an arbitrary generating parameter, leading in particular to the double-ended complex sequences $\{U_n(z)\}_{-\infty}^{\infty}$ and $\{V_n(z)\}_{-\infty}^{\infty}$. Later we shall make use of the numbers $U_n(z)$.

Let us now consider the 2-by-2 symmetric matrix

$$(2.10)\qquad \boldsymbol{M} = \begin{bmatrix} m & 1 \\ 1 & 0 \end{bmatrix}$$

which is governed by the parameter m and of which the eigenvalues are α_m and β_m. For n a nonnegative integer, it can be proved by induction [6] that

$$(2.11)\qquad \boldsymbol{M}^n = \begin{bmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{bmatrix}.$$

Also, the matrix $\boldsymbol{M}$ can obviously be extended to the case where the parameter m is arbitrary (e.g., complex).

2.2 A Cholesky Decomposition of the Matrix M: The Matrix M_k

Let us put

$$(2.12)\quad M_1 = M = \begin{bmatrix} m & 1 \\ 1 & 0 \end{bmatrix}$$

and decompose M_1 as

$$(2.13)\quad M_1 = T_1 T_1' = \begin{bmatrix} a_1 & 0 \\ c_1 & b_1 \end{bmatrix} \begin{bmatrix} a_1 & c_1 \\ 0 & b_1 \end{bmatrix},$$

where T_1 is a lower triangular matrix and the superscript (′) denotes transposition, so that T_1' is an upper triangular matrix. The values of the entries a_1, b_1, and c_1 of T_1 can be readily obtained by applying the usual matrix multiplication rule on the right-hand side of the matrix equation (2.13). In fact, the system

$$(2.14)\quad \begin{cases} a_1^2 = m \\ a_1 c_1 = 1 \\ b_1^2 + c_1^2 = 0 \end{cases}$$

can be written, whose solution is

$$(2.15)\quad \begin{cases} a_1 = \pm\sqrt{m} \\ c_1 = 1/a_1 \\ b_1 = \pm i c_1, \end{cases}$$

where $i = \sqrt{-1}$.

Any of these four solutions leads to a *Cholesky LR decomposition* [17] of the symmetric matrix M_1.

On the other hand, it is known [7] that a lower triangular matrix and an upper triangular matrix do not commute, so that the reverse product $T_1'T_1$ leads to the symmetric matrix M_2 which is similar to but different from M_1. If we take $b_1 = +ic_1$ [cf. (2.15)], we have

$$(2.16)\quad M_2 = \frac{1}{m}\begin{bmatrix} m^2 + 1 & i \\ i & -1 \end{bmatrix},$$

while, if we take $b_1 = -ic_1$, the off diagonal entries of M_2 become negative.

In turn, M_2 can be decomposed in a similar way, thus getting

$$M_2 = T_2 T_2' = \begin{bmatrix} a_2 & 0 \\ c_2 & b_2 \end{bmatrix} \begin{bmatrix} a_2 & c_2 \\ 0 & b_2 \end{bmatrix},$$

where

$$\begin{cases} a_2 = \pm\sqrt{(m^2 + 1)/m} \\ c_2 = 1/a_2 \\ b_2 = \pm i c_2. \end{cases}$$

The reverse product $T_2'T_2$ leads to a matrix M_3 with sign of the off diagonal entries depending on whether $b_2 = +ic_2$ or $-ic_2$ has been considered.

If we repeat such a procedure *ad infinitum*, we obtain the set $\{M_k\}_1^\infty$ of 2-by-2 symmetric matrices

(2.17) $$\mathbf{M}_k = \frac{1}{U_k}\begin{bmatrix} U_{k+1} & i^{k-1} \\ i^{k-1} & -U_{k-1} \end{bmatrix} \quad (k \geq 1).$$

Because of the ambiguity of signs that arises in the Cholesky factorization, (2.17) is not the only possible result of k applications of the LR algorithm to $\mathbf{M}$. However, the only other possible result differs from that shown in (2.17) only in the sign of the off diagonal entries. From here on, we will consider only the sequence $\{\mathbf{M}_k\}$ given by equation (2.17).

Since the matrices $\mathbf{M}_k$ are similar, their eigenvalues coincide. $\mathbf{M}_k$ tends to a diagonal matrix containing these eigenvalues (namely, α_m and β_m) as k tends to infinity.

To establish the general validity of (2.17), consider the Cholesky decomposition

$$\mathbf{M}_k = \frac{1}{U_k U_{k+1}}\begin{bmatrix} U_{k+1} & 0 \\ i^{k-1} & iU_k \end{bmatrix}\begin{bmatrix} U_{k+1} & i^{k-1} \\ 0 & iU_k \end{bmatrix}$$

where *Simson's formula*

(2.18) $$U_{k+1}U_{k-1} - U_k^2 = (-1)^k$$

has been invoked. Simson's formula may be quite readily established by using the Binet form (2.2) for U_n and the properties (2.4) of α_m and β_m. On the other hand, from (2.11) and (2.10), it is seen that

$$U_{k+1}U_{k-1} - U_k^2 = \det(\mathbf{M}^k) = (\det \mathbf{M})^k = (-1)^k.$$

Reversing the order of multiplication, we get

$$\frac{1}{U_k U_{k+1}}\begin{bmatrix} U_{k+1} & i^{k-1} \\ 0 & iU_k \end{bmatrix}\begin{bmatrix} U_{k+1} & 0 \\ i^{k-1} & iU_k \end{bmatrix} = \underbrace{\frac{1}{U_{k+1}}\begin{bmatrix} U_{k+2} & i^k \\ i^k & -U_k \end{bmatrix}}_{\text{[by (2.18)]}} \underset{\text{[by (2.17)]}}{=} \mathbf{M}_{k+1}.$$

Thus, if the matrix for $\mathbf{M}_k$ is valid, then so is the matrix for $\mathbf{M}_{k+1}$.

For convenience, $\mathbf{M}_k$ may be called the *Cholesky algorithm matrix of Fibonacci type* of order k.

Furthermore, if we apply the Cholesky algorithm to $\mathbf{M}^n$ [see (2.11)] rather than to $\mathbf{M}$, we obtain

(2.19) $$(\mathbf{M}^n)_k = \frac{1}{U_k}\begin{bmatrix} U_{k+n} & i^{k-1}U_n \\ i^{k-1}U_n & (-1)^n U_{k-n} \end{bmatrix} = \frac{1}{U_k}\begin{bmatrix} U_{n+k} & i^{k-1}U_n \\ i^{k-1}U_n & (-1)^{k-1}U_{n-k} \end{bmatrix}.$$

Observe that

(2.20) $$(\mathbf{M}^n)_k = (\mathbf{M}_k)^n = \mathbf{M}_k^n.$$

Validation of this statement may be achieved through an inductive argument. Assume (2.20) is true for some value of n, say N. Thus,

(2.21) $$(\mathbf{M}_k)^N = (\mathbf{M}^N)_k.$$

Then,

$$(\mathbf{M}_k)^{N+1} = \mathbf{M}_k(\mathbf{M}_k)^N \underset{\text{[by (2.21)]}}{=} \mathbf{M}_k(\mathbf{M}^N)_k = (\mathbf{M}^{N+1})_k$$

after a good deal of calculation, so that if (2.20) is true for N, it is also true for $N + 1$. In the calculations, it is necessary to derive certain identities among the U_n by using (2.2) and (2.4).

2.3 Functions of the Matrix xM_k^n

From the theory of functions of matrices [7], it is known that if f is a function defined on the spectrum of a 2-by-2 matrix $A = [a_{ij}]$ with distinct eigenvalues λ_1 and λ_2, then

$$f(A) = X = [x_{ij}] = c_0 I + c_1 A, \tag{2.22}$$

where I is the 2-by-2 identity matrix and the coefficients c_0 and c_1 are given by the solution of the system

$$\begin{cases} c_0 + c_1\lambda_1 = f(\lambda_1) \\ c_0 + c_1\lambda_2 = f(\lambda_2). \end{cases} \tag{2.23}$$

Solving (2.23) and using (2.22), after some manipulations we obtain

$$x_{11} = [(a_{11} - \lambda_1)f(\lambda_2) - (a_{11} - \lambda_2)f(\lambda_1)]/(\lambda_2 - \lambda_1) \tag{2.24}$$

$$x_{12} = a_{12}[f(\lambda_2) - f(\lambda_1)]/(\lambda_2 - \lambda_1) \tag{2.25}$$

$$x_{21} = a_{21}[f(\lambda_2) - f(\lambda_1)]/(\lambda_2 - \lambda_1) \tag{2.26}$$

$$x_{22} = [(a_{22} - \lambda_1)f(\lambda_2) - (a_{22} - \lambda_2)f(\lambda_1)]/(\lambda_2 - \lambda_1). \tag{2.27}$$

For x an arbitrary quantity, let us consider the matrix xM_k^n having eigenvalues

$$\begin{cases} \lambda_1 = x\alpha_m^n \\ \lambda_2 = x\beta_m^n \end{cases} \tag{2.28}$$

and let us find closed form expressions for the entries y_{ij} of

$$Y = [y_{ij}] = f(xM_k^n).$$

by (2.24)-(2.27), after some tedious manipulations involving the use of certain identities easily derivable from (2.2) and (2.3), we get

$$y_{11} = [\alpha_m^k f(x\alpha_m^n) - \beta_m^k f(x\beta_m^n)]/(\Delta_m U_k) \tag{2.29}$$

$$y_{12} = y_{21} = i^{k-1}[f(x\alpha_m^n) - f(x\beta_m^n)]/(\Delta_m U_k) \tag{2.30}$$

$$y_{22} = [\alpha_m^k f(x\beta_m^n) - \beta_m^k f(x\alpha_m^n)]/(\Delta_m U_k). \tag{2.31}$$

As an illustrative example, let f be the inverse function. Then, from (2.29)-(2.31) we obtain

$$(xM_k^n)^{-1} = \frac{(-1)^n}{xU_k}\begin{bmatrix} (-1)^{k-1}U_{n-k} & -i^{k-1}U_n \\ -i^{k-1}U_n & U_{n+k} \end{bmatrix} \tag{2.32}$$

$$= \frac{1}{x}M_k^{-n} \quad \text{[using (2.19) and (2.6)]}. \tag{2.33}$$

3. Some Applications of the Matrix M_k

In this and later sections some identities involving U_n and V_n are worked out as illustrations of the use of our Cholesky algorithm matrix of Fibonacci type M_k.

Example 1: From (2.19) we can write

$$M_k^k = \frac{1}{U_k}\begin{bmatrix} U_{2k} & i^{k-1}U_k \\ i^{k-1}U_k & 0 \end{bmatrix} = \begin{bmatrix} V_k & i^{k-1} \\ i^{k-1} & 0 \end{bmatrix} = R_k = [r_{ij}] \;\; (= [r_{ij}^{(1)}]), \tag{3.1}$$

whence

$$M_k^{nk} = R_k^n = [r_{ij}^{(n)}]. \tag{3.2}$$

Thus, $r_{11} = V_k$, $r_{12} = r_{21} = i^{k-1}$, $r_{22} = 0$. Take $r_{11}^{(0)} = 1$. By induction on n, with the aid of Pascal's formula for binomial coefficients, it can be proved that

$$\begin{cases} r_{11}^{(n)} = \sum_{j=0}^{[n/2]} (-1)^{j(k-1)} \binom{n-j}{j} V_k^{n-2j} \\ r_{12}^{(n)} = r_{21}^{(n)} = i^{k-1} r_{11}^{(n-1)} \\ r_{22}^{(n)} = (-1)^{k-1} r_{11}^{(n-2)} \qquad (n \ge 2). \end{cases} \tag{3.3}$$

On the other hand, the matrix M_k^{nk} can be expressed also [cf. (2.19)] as

$$M_k^{nk} = \frac{1}{U_k} \begin{bmatrix} U_{k(n+1)} & i^{k-1} U_{nk} \\ i^{k-1} U_{nk} & (-1)^{k-1} U_{k(n-1)} \end{bmatrix}. \tag{3.4}$$

Equating the upper left entries on the right-hand sides of (3.2) and (3.4), by (3.3) we obtain the identity

$$U_{k(n+1)}/U_k = \sum_{j=0}^{[n/2]} (-1)^{j(k-1)} \binom{n-j}{j} V_k^{n-2j}, \tag{3.5}$$

i.e., $U_k \mid U_{k(n+1)}$, as we would expect.

Furthermore, from (3.1) we can write

$$[(-i)^{k-1} M_k^k]^n = \begin{bmatrix} (-i)^{k-1} V_k & 1 \\ 1 & 0 \end{bmatrix}^n = Z_k^n = [z_{ij}^{(n)}], \tag{3.6}$$

where $Z_k = [z_{ij}]$ is an *extended M matrix* depending on the complex parameter

$$z = (-i)^{k-1} V_k(m). \tag{3.7}$$

From (2.11) we have

$$z_{12}^{(n)} = U_n(z), \tag{3.8}$$

and by equating $z_{12}^{(n)}$ and the upper right entry of $[(-i)^{k-1} M_k^k]^n$ obtained by (3.4) we can write

$$\begin{aligned} (-i)^{n(k-1)} i^{k-1} U_{nk}(m)/U_k(m) &= (-1)^{n(k-1)} i^{(n+1)(k-1)} U_{nk}(m)/U_k(m) \\ &= U_n(z). \end{aligned} \tag{3.9}$$

From (2.5) and (3.7) it can be verified that

$$U_n(z) = \begin{cases} U_n(V_k(m)) & (k \text{ odd}, n \text{ odd}) \\ (-1)^{(k-1)/2} U_n(V_k(m)) & (k \text{ odd}, n \text{ even}). \end{cases} \tag{3.10}$$

Therefore, from (3.9) and (3.10) we obtain the noteworthy identity

$$U_{nk}(m)/U_k(m) = U_n(V_k(m)) \qquad (k \text{ odd}) \tag{3.11}$$

which connects numbers defined by (2.1) having different generating parameters.

For instance,

$$(m^3 + 3m)^2 + 1 = m^6 + 6m^4 + 9m^2 + 1 = (m^8 + 7m^6 + 15m^4 + 10m^2 + 1)/(m^2 + 1)$$
$$= U_3(V_3(m)) \qquad = U_9(m)/U_3(m)$$

which simultaneously verifies (3.5) and (3.11).

Example 2: Following [2], from (2.19) we can write either

$$(3.12) \quad M_k^r M_k^s = \frac{1}{U_k^2}\begin{bmatrix} U_{r+k}U_{s+k} - (-1)^k U_r U_s & i^{k-1}[U_{r+k}U_s - (-1)^k U_r U_{s-k}] \\ i^{k-1}[U_{s+k}U_r - (-1)^k U_s U_{r-k}] & U_{r-k}U_{s-k} - (-1)^k U_r U_s \end{bmatrix}$$

or

$$(3.13) \quad M_k^{r+s} = \frac{1}{U_k}\begin{bmatrix} U_{r+s+k} & i^{k-1}U_{r+s} \\ i^{k-1}U_{r+s} & (-1)^{k-1}U_{r+s-k} \end{bmatrix}.$$

By equating the upper right entries on the right-hand sides of (3.12) and (3.13) we obtain

$$(3.14) \quad U_k U_{r+s} = U_{r+k}U_s - (-1)^k U_r U_{s-k}$$
$$= U_{s+k}U_r - (-1)^k U_s U_{r-k} \text{ also.}$$

4. Evaluation of Some Finite Series

In this section the sums of certain finite series involving U_n and V_n are found on the basis of some properties of the Fibonacci-type Cholesky algorithm matrix M_k.

It is readily seen from (2.17) and (2.19), with the aid of (2.1), that

$$(4.1) \quad M_k^2 = mM_k + I,$$

whence

$$(4.2) \quad M_k^{-1} = M_k - mI.$$

Moreover, using the identity

$$(4.3) \quad V_n U_p - U_{n+p} = (-1)^{p-1}U_{n-p},$$

which can be easily proved using (2.2) and (2.7), we can verify that

$$(4.4) \quad (xM_k^n - I)^{-1} = \frac{xM_k^n - (V_n x - 1)I}{(-1)^{n-1}x^2 + V_n x - 1}$$

where x is an arbitrary quantity subject by (2.28) to the restrictions

$$(4.5) \quad x \neq \begin{cases} 1/\alpha_m^n \\ 1/\beta_m^n. \end{cases}$$

A) From (4.1) we can write

$$(M_k^2 - I)^n = (mM_k)^n$$

and, therefore,

$$\sum_{j=0}^{n} (-1)^n \binom{n}{j} M_k^{2j} = m^n M_k^n,$$

whence, by (2.19), we obtain a set of identities which can be summarized by

$$(4.6) \qquad \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j}U_{2j+s} = m^n U_{n+s},$$

where n is a nonnegative integer and s an arbitrary integer. Replacing s by $s \pm 1$ in (4.6) and combining the results obtained, from (2.7) we have

$$(4.7) \qquad \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j}V_{2j+s} = m^n V_{n+s}.$$

Furthermore, following [13], from (4.1) we can write

$$(4.8) \qquad (mM_k + I)^n M_k^s = M_k^{2n+s}.$$

Equating appropriate entries on both sides of (4.8), with the aid of (2.19), we obtain

$$(4.9) \qquad \sum_{j=0}^{n} \binom{n}{j} m^j U_{j+s} = U_{2n+s},$$

whence, replacing s by $s \pm 1$ as earlier, we get

$$(4.10) \qquad \sum_{j=0}^{n} \binom{n}{j} m^j V_{j+s} = V_{2n+s}.$$

B) From (4.2) we can write

$$(M_k - mI)^n = (M_k^n)^{-1},$$

whence, by (2.19) and (2.32), after some manipulations, we obtain a set of identities which can be summarized by

$$(4.11) \qquad \sum_{j=0}^{n} (-1)^j m^{n-j}\binom{n}{j}U_{j+s} = (-1)^{s-1}U_{n-s}.$$

C) Finally, let us consider the identity

$$(4.12) \qquad (xA^n - I)\sum_{j=0}^{h} x^j A^{nj} = x^{h+1}A^{n(h+1)} - I,$$

which holds for any square matrix A. From (4.12) and (4.4) we can write, for the Cholesky algorithm matrix M_k of Fibonacci type,

$$(4.13) \qquad \sum_{j=0}^{h} x^j M_k^{nj} = (xM_k^n - I)^{-1}(x^{h+1}M_k^{n(h+1)} - I)$$

$$= \frac{xM_k^n - (V_n x - 1)I}{(-1)^{n-1}x^2 + V_n x - 1}(x^{h+1}M_k^{n(h+1)} - I)$$

$$= \frac{x^{h+2}M_k^{n(h+2)} - xM_k^n - x^{h+1}(V_n x - 1)M_k^{n(h+1)} + (V_n x - 1)I}{(-1)^{n-1}x^2 + V_n x - 1}.$$

After some manipulations involving the use of (4.3), from (4.13) and (2.19) we obtain a set of identities which can be summarized as

$$(4.14) \qquad \sum_{j=0}^{h} x^j U_{nj+s} = \frac{(-1)^{n-1}x^{h+2}U_{nh+s} + x^{h+1}U_{n(h+1)+s} - (-1)^s xU_{n-s} - U_s}{(-1)^{n-1}x^2 + V_n x - 1},$$

where n is a nonnegative integer and s is an arbitrary integer. Replacing s by $s \pm 1$ in (4.14), by (2.7) we can derive

$$(4.15) \qquad \sum_{j=0}^{h} x^j V_{nj+s} = \frac{(-1)^{n-1}x^{h+2}V_{nh+s} + x^{h+1}V_{n(h+1)+s} + (-1)^s xV_{n-s} - V_s}{(-1)^{n-1}x^2 + V_n x - 1}.$$

We point out that (4.14) and (4.15) obivously hold under the restrictions (4.5).

5. Evaluation of Some Infinite Series

In this section a method for finding the sums of certain infinite series involving U_n and V_n is shown which is based on the use of functions of the matrix xM_k^n (see Section 2.3).

Under certain restrictions, some sums can be worked out by using the results established in Section 4 above. For example, if

$$-1/\alpha_m^n < x < 1/\alpha_m^n, \tag{5.1}$$

we can take the limit of both sides of (4.14) and (4.15) as h tends to infinity thus getting

$$\sum_{j=0}^{\infty} x^j U_{nj+s} = \frac{(-1)^{s-1}xU_{n-s} - U_s}{(-1)^{n-1}x^2 + V_n x - 1}, \tag{5.2}$$

$$\sum_{j=0}^{\infty} x^j V_{nj+s} = \frac{(-1)^s xV_{n-s} - V_s}{(-1)^{n-1}x^2 + V_n x - 1}. \tag{5.3}$$

5.1 Use of Certain Functions of xM_k^n

Following [6], we consider the power series expansion of $\exp(xM_k^n)$ [7],

$$Y = \exp(xM_k^n) = \sum_{j=0}^{\infty} \frac{x^j M_k^{jn}}{j!} \tag{5.4}$$

and the closed form expressions of the entries y_{ij} of Y derivable from (2.29)-(2.31) by letting f be the exponential function. Equating y_{ij} and the corresponding entry of Y on the right-hand side of (5.4), from (2.19) we obtain the identities

$$\sum_{j=0}^{\infty} \frac{x^j U_{jn+k}}{j!} = [\alpha_m^k \exp(x\alpha_m^n) - \beta_m^k \exp(x\beta_m^n)]/\Delta_m, \tag{5.5}$$

$$\sum_{j=0}^{\infty} \frac{x^j U_{jn}}{j!} = [\exp(x\alpha_m^n) - \exp(x\beta_m^n)]/\Delta_m, \tag{5.6}$$

$$\sum_{j=0}^{\infty} \frac{x^j U_{jn-k}}{j!} = (-1)^{k-1}[\alpha_m^k \exp(x\beta_m^n) - \beta_m^k \exp(x\alpha_m^n)]/\Delta_m, \tag{5.7}$$

which, by using the identity $(-1)^{k-1}\alpha_m^{-k} = -\beta_m^k$ [see (2.4)], can be summarized as

$$\sum_{j=0}^{\infty} \frac{x^j U_{jn+s}}{j!} = [\alpha_m^s \exp(x\alpha_m^n) - \beta_m^s \exp(x\beta_m^n)]/\Delta_m, \tag{5.8}$$

where n is a nonnegative integer and s is an arbitrary integer.

From (5.8), (2.7), and (2.3) we can readily derive

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{x^j V_{jn+s}}{j!} &= [\alpha_m^{s-1}\exp(x\alpha_m^n)(1 + \alpha_m^2) - \beta_m^{s-1}\exp(x\beta_m^n)(1 + \beta_m^2)]/\Delta_m \\ &= \alpha_m^{s-1}\exp(x\alpha_m^n)(\Delta_m + m)/2 - \beta_m^{s-1}\exp(x\beta_m^n)(\Delta_m - m)/2 \\ &= \alpha_m^s \exp(x\alpha_m^n) + \beta_m^s \exp(x\beta_m^n).\end{aligned} \tag{5.9}$$

By considering power series expansions [1], [16], [7] of other functions of the matrix xM_k^n, the above presented technique allows us to evaluate a very large amount of infinite series involving numbers U_n and V_n. We confine ourselves to showing an example derived from the expansion of $\tan^{-1} y$ (see [1] and [7, p. 113].

Under the restriction

$$(5.10) \quad -1/\alpha_m^n \le x \le 1/\alpha_m^n,$$

we have

$$(5.11) \quad \sum_{j=1}^{\infty} \frac{(-1)^{j+1} x^{2j-1} U_{n(2j-1)+s}}{2j-1} = [\alpha_m^s \tan^{-1}(x\alpha_m^n) - \beta_m^s \tan^{-1}(x\beta_m^n)]/\Delta_m.$$

6. Conclusion

The identities established in this paper represent only a small sample of the possibilities available to us. We believe that the Cholesky decomposition matrix M_k is a useful tool for discovering many more identities. Further investigations into the properties of matrices of this type are warranted.

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