ZECKENDORF NUMBER SYSTEMS AND ASSOCIATED PARTITIONS

Clark Kimberling
University of Evansville, Evansville, IN 47702
(Submitted April 1989)

The binary number system lends itself to unrestricted ordered partitions, as indicated in Table 1.

TABLE 1. The Binary Case

n	Binary Representation	k	Associated Partition of k
1	1	1	1
2	10	2	2
3	11	2	11
4	100	3	3
5	101	3	21
6	110	3	12
7	111	3	111
8	1000	4	4
9	1001	4	31
10	1010	4	22
11	1011	4	211
12	1100	4	13
13	1101	4	121
14	1110	4	112
15	1111	4	1111
16	10000	5	8

Note that the partitions of $k = 4$, ranging from 4 to 1111, are in one-to-one correspondence with the integers from 8 to 15, for a total of 8 partitions. Similarly, there are 16 partitions of 5, 32 of 6, and generally, 2^{k-1} partitions of k. These are in one-to one correspondence with the binary representations of length k.

It is well known (Zeckendorf [1]) that the Fibonacci numbers

$$F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8,\ F_7 = 13,\ \ldots$$

serve as a basis for another zero-one number system, depending on unique sums of nonconsecutive Fibonacci numbers. These sums are often called Zeckendorf representations (see Table 2). The partitions of k that appear in this scheme are those in which only the last term can equal 1; that is,

$$k = r_1 + r_2 + \cdots + r_j,\text{ where } r_i \geq 2 \text{ for } i < j \text{ and } r_j \geq 1.$$

Table 2 suggests that, for any k, the number of partitions in which 1 is allowed only in the last place is the Fibonacci number F_k (e.g., 34 - 21 = 13 partitions of 7, ranging from 7 to 2221). This is nothing new, since the number of zero-one sequences of length k beginning with 1 and having no two consecutive 1's is well known to be F_k. It is less well known that these zero-one sequences correspond to partitions.

TABLE 2. The Zeckendorf Case

n	Zeckendorf Representation	Zero-One Representation	k	Associated Partition of k
1	1	1	1	1
2	2	10	2	2
3	3	100	3	3
4	3 + 1	101	3	21
5	5	1000	4	4
6	5 + 1	1001	4	31
7	5 + 2	1010	4	22
8	8	10000	5	5
⋮				
21	21	1000000	7	7
22	21 + 1	1000001	7	61
23	21 + 2	1000010	7	52
24	21 + 3	1000100	7	43
25	21 + 3 + 1	1000101	7	421
⋮				
32	21 + 8 + 3	1010100	7	223
33	21 + 8 + 3 + 1	1010101	7	2221
34	34	10000000	8	8

Here is a summary of the observations from Tables 1 and 2. The first-order recurrence sequence 1, 2, 4, 8, ... serves as a basis for unrestricted partitions, and the second-order recurrence sequence 1, 2, 3, 5, 8, ... serves as a basis for somewhat restricted partitions.

The purpose of this article is to extend these results to higher-order sequences, their zero-one number systems, and associated partitions. To this end, and for the remainder of the article, let m be an arbitrary fixed integer greater than 2.

Define a sequence $\{s_i\}$ inductively as follows:

$$s_i = 1 \qquad \text{for } i = 1, 2, \ldots, m,$$

$$s_i = s_{i-1} + s_{i-m} \quad \text{for } i = m+1, m+2, \ldots .$$

Theorem 1: Every positive integer n is uniquely a sum

$$s_{i_1} + s_{i_2} + \cdots + s_{i_v}, \text{ where } i_t - i_u \geq m \text{ whenever } t > u.$$

Proof: The first m positive integers are one-term sums. Suppose, for $h \geq m + 1$, that the statement of the theorem holds for all $n \leq h - 1$. Let i_1 be the greatest i for which $s_i \leq h$. If $h - s_{i_1} = 0$, then the required sum is s_{i_1} itself.

Otherwise, $h - s_{i_1}$ is, by the induction hypothesis, uniquely a sum $s_{i_2} + \cdots + s_{i_v}$ of the required sort, so that

$$(1) \qquad h = s_{i_1} + s_{i_2} + \cdots + s_{i_v}.$$

Suppose $i_1 - i_2 \leq m - 1$. Then

$$h \geq s_{i_1} + s_{i_2} \geq s_{i_1} + s_{i_1 - m+1} = s_{i_1+1},$$

contrary to our choice of i_1 as the greatest i for which $h \geq s_i$.

Therefore, the sum in (1) has $i_t - i_u \geq m$ whenever $t > u$, and this sum is clearly unique with respect to this property. By the principle of mathematical induction, the proof of the theorem is finished.

Theorem 1 shows that the sequence $\{s_i\}$ serves as a basis for a "skip $m - i$ number system" analogous to the Zeckendorf, or Fibonacci, number system. The latter could be called the "skip 1 number system."

Examples: In the skip 1 system:

$$\begin{aligned} 31 &= 21 + 8 + 2 &&= 1010010 \\ 32 &= 21 + 8 + 3 &&= 1010100 \\ 33 &= 21 + 8 + 3 + 1 &&= 1010101 \\ 34 &= 34 &&= 1000000 \end{aligned}$$

In the skip 2 system:

$$\begin{aligned} 57 &= 41 + 13 + 3 &&= 1001000100 \\ 58 &= 41 + 13 + 4 &&= 1001001000 \\ 59 &= 41 + 13 + 4 + 1 &&= 1001001001 \\ 60 &= 60 &&= 10000000000 \end{aligned}$$

We turn now to partitions. For a quick glimpse of what is coming, notice that the zero-one representations for 57, 58, and 59, just above, lend themselves naturally to the partitions 343, 334, and 3331 of the integer 10.

In general, in the $m - 1$ system, for a given positive integer k, the digit one occurs at and only at places $i_1, i_2, \ldots, i_v$, where $k = s_{i_1} + s_{i_2} + \cdots + s_{i_v}$, and each pair of ones are separated by at least $m - 1$ zeros; therefore, to each k there is a unique ordered v-tuple of integers r_i defined by

$$(2) \qquad \begin{cases} r_1 = i_1, \text{ if } v = 1, \\ r_u = i_u - i_{u+1} \text{ for } u = 1, 2, \ldots, v - 1, \text{ if } v > 1 \text{ and } s_{i_v} \geq m, \\ r_u = i_u - i_{u+1} \text{ for } u = 1, 2, \ldots, v - 1 \text{ and } r_v = i_v, \\ \qquad\qquad \text{if } v > 1 \text{ and } s_{i_v} \leq m - 1. \end{cases}$$

We summarize these observations in Theorem 2.

Theorem 2: Let k be a positive integer, let $S_k = \{s_k, s_k + 1, \ldots, s_{k+1} - 1\}$, and let P_k be the set of partitions $r_1, r_2, \ldots, r_v$ of k that satisfy $r_v \geq 1$ and $r_i \geq m$ for $i = 1, 2, \ldots, m - 1$. Then equations (2) define a one-to-one correspondence between S_k and P_k, so that the number $p(k)$ of partitions in P_k is s_{k-m-1}.

Now for any positive integer k, and for $j = 1, 2, \ldots, m$, let $p(k, j)$ be the number of partitions $r_1, r_2, \ldots, r_v$ of k for which $r_v = j$ and $r_i \geq m$ for $1 = 1, 2, \ldots, v - 1$. As in Theorem 2, let $p(k)$ be the number of partitions of k for which $r_v \geq 1$ and $r_i \geq m$ for $i = 1, 2, \ldots, v - 1$. Let $q(k)$ be the number of partitions of k for which $r_i \geq m$ for *all* indices $i = 1, 2, \ldots, v - 1, v$.

Lemma 1:

$$p(k, j) = \begin{cases} 1 & \text{if } k = j \leq m, \\ 0 & \text{if } k \leq m,\ j \leq m, \text{ and } k \neq j. \end{cases}$$

Proof: For any given $k \leq m$, the partition of k is the number k by itself, so that $p(k, k) = 1$. Clearly, $p(k, j) = 0$ for $k \neq j$ since, in this case, no partition of the form described is possible.

Lemma 2: Suppose $i \leq j \leq m$. Then $p(k, j) = p(k - 1, j) + p(k - m, j)$ for $k = m + 1, m + 2, \ldots$.

Proof: Assume $k \geq m + 1$. Each of the $p(k - 1, j)$ partitions $r_1, r_2, \ldots, r_{v-1}, j$ of $k - 1$ yields a partition $r_1 + 1, r_2, \ldots, r_{v-1}, j$ of k. Moreover, $r_1 + 1 \geq m + 1$, so that every partition of k having first term $\geq m + 1$ corresponds in this manner to a partition of $k - 1$.

Each of the $p(k - m, j)$ partitions $r_2, r_3, \ldots, j$ of $k - m$ yields a partition $m, r_2, r_3, \ldots, j$ of k. Moreover, every partition of k having first term

m corresponds in this manner to a partition of $k - m$.

Since $p(k, j)$ counts partitions having first term $\geq m$, a proof that

$$p(k, j) = p(k - 1, j) + p(k - m, j)$$

is finished.

Theorem 3: Suppose k is a positive integer. The number $q(k)$ of partitions $r_1, r_2, \ldots, r_v$ of k having $r_i \geq m$ for $i = 1, 2, \ldots, v$ is given by the m^{th}-order linear recurrence $q(k) = q(k - 1) + q(k - m)$ for $k = m + 1, m + 2, \ldots,$ where $q(j) = 0$ for $j = 1, 2, \ldots, m - 1$, and $q(m) = 1$.

Proof: The assertion follows directly from Lemma 2, since

$$q(k) = p(k) - \sum_{j=1}^{m-1} p(k, j).$$

Reference

1. E. Zeckendorf. "Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas." *Bull. Soc. Royale Sci. Liège 41* (1972):179-82.

*****