COMBINATORIAL REPRESENTATION OF GENERALIZED FIBONACCI NUMBERS*

Shmuel T. Klein
Bar-Ilan University, Ramat-Gan 52 900, Israel
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1. Introduction

A well-known combinatorial formula for the Fibonacci numbers F_n, defined by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$, is

$$(1) \qquad \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} = F_{n+1} \text{ for } n \ge 0,$$

which can be shown by induction (see, for example, Knuth [11, Ex. 1.2.8-16]). The following proof, however, is easily generalizable to various other recursively defined sequences of integers.

The Fibonacci numbers $\{F_2, F_3, \ldots\}$ are the basis elements of the *binary Fibonacci numeration system* (see [11, Ex. 1.2.8-34] or Fraenkel [7]). Every integer K in the range $0 \le K < F_{n+1}$ has a unique binary representation of $n - 1$ bits, $k_{n-1}k_{n-2} \ldots k_1$, such that

$$K = \sum_{i=1}^{n-1} k_i F_{i+1}$$

and such that there are no adjacent 1's in this representation of K (see Zeckendorf [17]). It follows that, for $n \ge 1$, F_{n+1} is the number of binary strings of length $n - 1$ having no adjacent 1's. The number of these strings with precisely i 1's, $0 \le i \le \lfloor n/2 \rfloor$, is evaluated using the fact that the number of possibilities to distribute a indistinguishable objects into $b + 1$ disjoint sets, of which $b - 1$ should contain at least one element, is $\binom{a+1}{b}$ (see Feller [6, Sec. II.5]). In our case, there are $n - 1 - i$ zeros to be partitioned into $i + 1$ runs, of which the $i - 1$ runs delimited on both sides by 1's should be nonempty; the number of these strings is therefore $\binom{n-i}{i}$.

In a similar way, counting strings of certain types, Philippou and Muwafi [15] derived a representation of Fibonacci numbers of order m, with $m \ge 2$, as a sum of multinomial coefficients; their formula coincides with that presented earlier by Miles [13].

The properties of the representation of integers in Fibonacci-type numeration systems were used by Kautz [10] for synchronization control. More recently, they were investigated in Pihko [16] and exploited in various applications, such as the compression of large sparse bit-strings (see Fraenkel and Klein [8]), the robust transmission of binary strings in which the length is in an unknown range (see Apostolico and Fraenkel [3]), and the evaluation of the potential number of phenotypes in a model of biological processing of genetic information based on the majority rule (see Agur, Fraenkel, and Klein [1]). In the present work, the properties of numeration systems are used to generate new combinatorial formulas. In the next section, this is done for the sequence based on the recurrence $a_i = a_{i-1} + a_{i-m}$, for some $m \ge 2$, which appears in certain applications to encoding algorithms for CD-ROM. Section 3

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deals with other generalizations of Fibonacci numbers, namely, sequences based on the recurrences $u_i = mu_{i-1} + u_{i-2}$ for $m \geq 1$, or $v_i = mv_{i-1} - v_{i-2}$ for $m \geq 3$, which are special cases of the sequences investigated by Horadam [9]. For certain values of m and with appropriate initial values, these two recurrence relations generate the subsequences of every k^{th} Fibonacci number, for all $k \geq 1$. For further details on the properties of numeration systems, the reader is referred to [7].

2. A Generalization of Fibonacci Numbers

Given a constant integer $m \geq 2$, consider the sequence defined by

$$A_n^{(m)} = n - 1 \qquad \text{for } 1 < n \leq m + 1,$$

$$A_n^{(m)} = A_{n-1}^{(m)} + A_{n-m}^{(m)} \text{ for } n > m + 1.$$

In particular, $F_n \equiv A_n^{(2)}$ are the standard Fibonacci numbers. It follows from [7, Th. 1] that, for fixed m, the numbers $\{A_2^{(m)}, A_3^{(m)}, \ldots\}$ are the basis elements of a binary numeration system with the following property: every integer K in the range $0 \leq K < A_{n+1}^{(m)}$ has a unique binary representation of $m - 1$ bits, $k_{n-1}k_{n-2}\ldots k_1$, such that

$$K = \sum_{i=1}^{n-1} k_i A_{i+1}^{(m)}$$

and such that there are at least $m - 1$ zeros between any two 1's in this representation of K. Hence, for $n \geq 1$, $A_{n+1}^{(m)}$ is the number of binary strings of length $n - 1$ having this property.

For $n = 2$, we again get the property that there are no adjacent ones in the binary representation.

An interesting application of the sequence $A_n^{(m)}$ is to analyze encoding methods for certain optical discs. A CD-ROM (compact disc-read only memory) is an optical storage medium able to store large amounts of digital data (about 550 MB or more). The information, represented by a spiral of almost two billion tiny pits separated by spaces, is molded onto the surface of the disc. A digit 1 is represented by a transition from a pit to a space or from a space to a pit, and the length of a pit or space indicates the number of zeros. Due to the physical limitations of the optical devices, the lengths of pits and spaces are restricted, implying that there are at least two 0's between any two 1's (for details, see, for example, Davies [4]): this is the case $m = 3$ of our sequence above. It follows that if we want to encode a standard ASCII byte (256 possibilities), we need at least 14 bits, which corresponds to $A_{16}^{(3)} = 277$. In fact, there is an additional restriction that no more than 11 consecutive zeros are allowed, which disqualifies 6 of the 277 strings, but 14 bits are still enough; indeed, the code used for CD-ROM is called EFM (eight to fourteen modulation).

We now derive a combinatorial formula for $A_{n+1}^{(m)}$. First, note that $A_{n+1}^{(m)}$ is also the number of binary strings of length $n + m - 2$, with zeros in its $m - 1$ rightmost bits, such that every 1 is immediately followed by $m - 1$ zeros. Let k be the number of 1's in such a string, so that k can take values from 0 to $\lfloor (n + m - 2)/m \rfloor$. We now consider the string consisting of elements of two types: blocks of the form 10...0 ($m - 1$ zeros) and single zeros; there are k elements of the first type and $(n + m - 2) - km$ of the second, which can be arranged in

$$\binom{n + m - 2 - (m - 1)k}{k}$$

ways. Thus, we have the following formula, holding for $m \geq 2$ and $n \geq 1$:

$$(2) \qquad \sum_{k=0}^{\lfloor (n+m-2)/m \rfloor} \binom{n+m-2-(m-1)k}{k} = A_{n+1}^{(m)}.$$

For $m = 2$, (2) reduces to formula (1). Using the example mentioned above for EFM codes, setting $m = 3$ and $n = 15$, we get:

$$\binom{16}{0} + \binom{14}{1} + \binom{12}{2} + \binom{10}{3} + \binom{8}{4} + \binom{6}{5}$$
$$= 1 + 14 + 66 + 120 + 70 + 6 = 277 = A_{16}^{(3)}.$$

3. Regular Fibonacci Subsequences

Let L_n be the n^{th} Lucas number, defined by $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. The standard extension to negative indices sets

$$L_{-n} = (-1)^n L_n \quad \text{and} \quad F_{-n} = (-1)^{n+1} F_n \text{ for } n \geq 1.$$

We are interested in the *regular subsequences* of the Fibonacci sequence obtained by scanning the latter in intervals of size k, i.e., the sequences $\{F_{kn+j}\}_{n=-\infty}^{\infty}$ for all constant integers $k \geq 2$ and $0 \leq j < k$. The following identity, which is easily checked and apparently due to Lucas (see Dickson [5, p. 395]), shows that all the subsequences with the same interval size k satisfy a simple recurrence relation: for all (positive, null, or negative) integers k and n,

$$(3) \qquad F_n = L_k F_{n-k} + (-1)^{k+1} F_{n-2k}.$$

It follows that all regular subsequences of the Fibonacci numbers can be generated by a recurrence relation of the type $u_i = mu_{i-1} \pm u_{i-2}$, for certain values of m, and with appropriate initial conditions. We now apply the above techniques to obtain combinatorial representations of these number sequences.

For fixed $m \geq 3$, define a sequence of integers by

$$U_0^{(m)} = 0, \; U_1^{(m)} = 1, \text{ and } U_n^{(m)} = mU_{n-1}^{(m)} - U_{n-2}^{(m)} \text{ for } n \geq 2.$$

The numbers $\{U_1^{(m)}, U_2^{(m)}, \ldots\}$ are the basis elements of an m-ary numeration system: every integer K in the range $0 \leq K < U_n^{(m)}$ has a representation of $n - 1$ "m-ary digits," $k_{n-1}k_{n-2}\ldots k_1$, with $0 \leq k_i \leq m - 1$, such that

$$K = \sum_{i=1}^{n-1} k_i U_i^{(m)};$$

this representation is unique if the following property holds: if, for some $1 \leq i < j \leq n - 1$, k_i and k_j both assume their maximal value $m - 1$, then there exists an index s satisfying $i < s < j$, for which $k_s \leq m - 3$ (see [7, Th. 4]). In particular, for $m = 3$, we get a ternary system based on the even-indexed Fibonacci numbers $\{1, 3, 8, 21, \ldots\}$, and in the representation of any integer using this sequence as basis elements, there is at least one zero between any two 2's.

For general m, we have that $U_n^{(m)}$ is the number of m-ary strings of length $n - 1$, such that, between any two $(m - 1)$'s, there is at least one of the digits $0, \ldots, (m - 3)$. For a given m-ary string A of length $n - 1$, let j_i be the number of i's in A, $0 \leq i \leq m - 1$, thus, $0 \leq j_i < n$ and

$$\sum_{i=0}^{m-1} j_i = n - 1.$$

To construct an m-ary string satisfying the condition, we first arrange the digits 0, ..., $(m - 3)$ in any order, which can be done in

$$\binom{\sum_{i=0}^{m-3} j_i}{j_0,\ j_1,\ \ldots,\ j_{m-3}}$$

ways. Then the j_{m-1} $(m - 1)$'s have to be interspersed, with no two of them adjacent. In other words, the $\sum_{i=0}^{m-3} j_i$ smaller digits, which are now considered indistinguishable, are partitioned into $j_{m-1} + 1$ sets, of which at least $j_{m-1} - 1$ should be nonempty; there are

$$\binom{n - j_{m-2} - j_{m-1}}{j_{m-1}}$$

possibilities for this partition. Finally, the $(m - 2)$'s can be added anywhere, in

$$\binom{n - 1}{j_{m-2}}$$

ways. This yields the followinig formula, holding for $m \geq 3$ and $n \geq 1$:

$$\sum_{\substack{j_0, \ldots, j_{m-1} \geq 0 \\ j_0 + \cdots + j_{m-1} = n - 1}} \binom{n - 1 - j_{m-2} - j_{m-1}}{j_0,\ j_1,\ \ldots,\ j_{m-3}} \binom{n - 1}{j_{m-2}} \binom{n - j_{m-2} - j_{m-1}}{j_{m-1}} = U_n^{(m)}. \tag{4}$$

Using the fact that for integers a and b, $\binom{a}{b} = 0$ if $0 \leq a < b$, there is no need to impose further restrictions on the indices, but the rightmost binomial coefficient in (4) implies that j_{m-1} varies in fact in the range $0 \leq j_{m-1} \leq \lceil (n - 1)/2 \rceil$. The sequence $(U_n^{(m)})$ corresponds to the sequence $(w_n(0, 1; m, 1))$ studied by Horadam [9], but formula (4) is different from Horadam's identity (3.20).

Remark: Noting that the definition and the multinomial expansion of the multivariate Fibonacci polynomials of order k $\{H_n^{(k)}(x_1, \ldots, x_k)\}$ of Philippou and Antzoulakos [14] may be trivially extended to $x_j \in R$ $(j = 1, \ldots, k)$, we readily get the following alternative to (4), namely,

$$U_n^{(m)} = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n - 1 - j}{j} (-1)^j m^{n-1-2j}, \quad m \geq 3,\ n \geq 1,$$

since $\{U_n^{(m)}\} = \{H_n^{(2)}(m, -1)\}$ $(m \geq 3,\ n \geq 1)$.

From (3), we know that the regular subsequence $\{F_{k(n-1)+j}\}_{n=0}^{\infty}$ of the Fibonacci numbers, for constant even $k \geq 2$ and $0 \leq j < k$, is obtained by the same recurrence relation as the sequence $\{U_n^{(L_k)}\}_{n=0}^{\infty}$, with the difference that the first two elements (indexed 0 and 1) must be defined as F_{-k+j} and F_j instead of 0 and 1. Thus, we can express the Fibonacci subsequences with even interval size in terms of $U^{(m)}$:

Theorem 1: For any even constant $k \geq 2$ and any constant $0 \leq j < k$, the following identity holds for all $n \geq 1$:

$$F_{k(n-1)+j} = F_j U_n^{(L_k)} - F_{-k+j} U_{n-1}^{(L_k)}. \tag{5}$$

Proof: By induction on n. For $n = 1$,

$$F_j = F_j \times 1 - F_{-k+j} \times 0.$$

For $n = 2$,

$$F_{k+j} = L_k F_j + (-1)^{k+1} F_{-k+j} \quad \text{by (3)},$$

but since k is even, the right-hand side can be rewritten as

$$F_j U_2^{(L_k)} - F_{-k+j} U_1^{(L_k)}.$$

Suppose the identity holds for all integers $\le n$. Then

$$\begin{aligned} F_{kn+j} &= L_k F_{k(n-1)+j} - F_{k(n-2)+j} \\ &= L_k [F_j U_n^{(L_k)} - F_{-k+j} U_{n-1}^{(L_k)}] - F_j U_{n-1}^{(L_k)} + F_{-k+j} U_{n-2}^{(L_k)} \\ &= F_j U_{n+1}^{(L_k)} - F_{-k+j} U_n^{(L_k)}, \end{aligned}$$

so the identity holds also for $n + 1$, and therefore for all $n \ge 1$. □

In particular, for $j = 0$ and $k = 2$, we get the numbers $F_{2(n-1)}$, $n = 1, 2, \ldots$, which are the even-indexed Fibonacci numbers, and correspond by (5) to

$$U_{n-1}^{(L_2)} = U_{n-1}^{(3)}.$$

For $m = L_2 = 3$, the multinomial coefficient in (4) reduces to $\binom{j_0}{j_0} = 1$, and the equivalent of (4) can therefore be rewritten as:

$$\sum_{j_2=0}^{\lceil (n-1)/2 \rceil} \sum_{j_0=\max(0,\, j_2-1)}^{n-1-j_2} \binom{n-1}{j_0+j_2}\binom{j_0+1}{j_2} = \underline{U_n^{(3)} = F_{2n}}.$$

For example, for $n = 4$, we get:

$$\begin{aligned} &\binom{3}{0}\binom{1}{0} + \binom{3}{1}\binom{2}{0} + \binom{3}{2}\binom{3}{0} + \binom{3}{3}\binom{4}{0} + \binom{3}{1}\binom{1}{1} + \binom{3}{2}\binom{2}{1} + \binom{3}{3}\binom{3}{1} + \binom{3}{3}\binom{2}{2} \\ &= 1 + 3 + 3 + 1 + 3 + 6 + 3 + 1 = 21 = U_4^{(3)} = F_8. \end{aligned}$$

For fixed $m \ge 1$, define a sequence of integers by

$$\underline{V_0^{(m)} = 1,\ V_1^{(m)} = 1, \text{ and } V_n^{(m)} = mV_{n-1}^{(m)} + V_{n-2}^{(m)} \text{ for } n \ge 2.}$$

The numbers $\{V_1^{(m)}, V_2^{(m)}, \ldots\}$ are the basis elements of an $(m + 1)$-ary numeration system with the following property: every integer K in the range $0 \le K < V_n^{(m)}$ has a unique representation of $n - 1$ "$(m + 1)$-ary digits," $k_{n-1}k_{n-2}\ldots k_1$, with $0 \le k_i \le m$, such that

$$K = \sum_{i=1}^{n-1} k_i V_i^{(m)}$$

and such that, for $i \ge 1$, if k_{i+1} assumes its maximal value m, then $k \ = 0$ (see [7, Th. 3]). In particular, for $m = 1$, we get the binary numeration system based on the Fibonacci sequence and the condition that there are no adjacent 1's.

For general m, we have that $V_n^{(m)}$ is the number of $(m + 1)$-ary strings of length $n - 1$, such that when scanning the string from left to right, every appearance of the digit m, unless it is in the last position, is immediately followed by a digit 0. Special treatment of the rightmost digit is avoided by noting that $V_n^{(m)}$ is also the number of $(m + 1)$-ary strings of length n, with 0 in its rightmost position, and where each digit m is followed by a digit 0. For a given $(m + 1)$-ary string A of length n, let j_i be the number of i's in A, $0 \le i \le m$, thus $0 \le j_i \le n$ and

$$\sum_{i=0}^{m} j_i = n.$$

To construct an $(m + 1)$-ary string satisfying the condition, distribute the 0's in the spaces between the m's, such that every m is followed by at least one 0. In other words, the j_0 zeros have to be partitioned into $j_m + 1$ sets, of which at least j_m should be nonempty; there are $\binom{j_0}{j_m}$ possibilities for this partition.

We now consider the string obtained so far as consisting of j_0 units, where each unit is either one of the j_m pairs "$m0$" or one of the remaining $j_0 - j_m$ single zeros. The digits 1, ..., $(m - 1)$ are then to be distributed in the spaces between these units, including the space preceding the first unit, but not after the last unit, because the rightmost position must be 0. First the digits 1, ..., $(m - 1)$ are arranged in any order, which can be done in

$$\binom{\sum_{i=1}^{m-1} j_i}{j_1, \ldots, j_{m-1}}$$

ways; finally, these $\sum_{i=1}^{m-1} j_i$ digits, which are considered indistinguishable, are partitioned into j_0 sets, which can be done in

$$\binom{\sum_{i=0}^{m-1} j_i - 1}{j_0 - 1} = \binom{n - 1 - j_m}{n - j_0 - j_m}$$

ways. Summarizing, we get, for $m \geq 1$ and $n \geq 1$:

$$\text{(6)} \qquad \sum_{\substack{j_0 > 0,\ j_1, \ldots,\ j_m \geq 0 \\ j_0 + \cdots + j_m = n}} \binom{j_0}{j_m} \binom{n - 1 - j_m}{j_0 - 1,\ j_1,\ \ldots,\ j_{m-1}} = V_n^{(m)}.$$

For $m = 1$, the multinomial coefficient is $\binom{j_0 - 1}{j_0 - 1} = 1$, and we again get (1). For $m = 2$, the sequence $\{V_n^{(2)}\}$ is $\{1, 3, 7, 17, \ldots\}$, and the ternary numeration system based on this sequence is the system which yielded the best compression results in [8]. The sequence $\{V_n^{(m)}\}$ corresponds to $\{w_n(1, 1; m, -1)\}$ in [9], but again the combinatorial representation (6) is different from Horadam's formula (3.20). For $m = 2$, (6) reduces to:

$$\sum_{j_2 = 0}^{\lceil (n-1)/2 \rceil} \sum_{j_0 = \max(1,\ j_2)}^{n - j_2} \binom{j_0}{j_2} \binom{n - 1 - j_2}{j_0 - 1} = V_n^{(2)}.$$

For example, for $n = 3$, we get:

$$\binom{1}{0}\binom{2}{0} + \binom{2}{0}\binom{2}{1} + \binom{3}{0}\binom{2}{2} + \binom{1}{1}\binom{1}{0} + \binom{2}{1}\binom{1}{1}$$
$$= 1 + 2 + 1 + 1 + 2 = 7 = V_3^{(2)}.$$

Returning to the regular subsequences of the Fibonacci numbers, we still need a combinatorial representation of the subsequences with odd interval size k, which by (3) satisfy the same recurrence relation as $V_n^{(L_k)}$, but possibly with other initial values. The counterpart of Theorem 1 for the odd intervals is:

Theorem 2: For any odd constant $k \geq 1$ and any constant $0 \leq j < k$, the following identity holds for all $n \geq 1$:

$$\text{(7)} \qquad F_{k(n-1)+j} = F_j V_n^{(L_k)} + (F_{-k+j} - F_j) \sum_{i=1}^{n-1} (-1)^{i+1} V_{n-i}^{(L_k)}.$$

Proof: By induction on n. For $n = 1$,

$$F_j = F_j \times 1 + (F_{-k+j} - F_j) \times 0.$$

For $n = 2$,

$$\begin{aligned} F_{k+j} &= L_k F_j + F_{-k+j} = F_j (L_k + 1) + (F_{-k+j} - F_j) \\ &= F_j V_2^{(L_k)} + (F_{-k+j} - F_j) V_1^{(L_k)}. \end{aligned}$$

Suppose the identity holds for all integers $\leq n$. Then, denoting the constant $(F_{-k+j} - F_j)$ by α,

$$\begin{aligned}F_{kn+j} &= L_k F_{k(n-1)+j} + F_{k(n-2)+j}\\ &= L_k\left[F_j V_n^{(L_k)} + \alpha\sum_{i=1}^{n-1}(-1)^{i+1}V_{n-i}^{(L_k)}\right] + F_j V_{n-1}^{(L_k)} + \alpha\sum_{i=1}^{n-2}(-1)^{i+1}V_{n-1-i}^{(L_k)}\\ &= F_j V_{n+1}^{(L_k)} + \alpha\sum_{i=1}^{n-2}(-1)^{i+1}[L_k V_{n-i}^{(L_k)} + V_{n-1-i}^{(L_k)}] + \alpha L_k(-1)^n V_1^{(L_k)}.\end{aligned}$$

But the last term is

$$\alpha(-1)^n L_k = \alpha(-1)^n(V_2^{(L_k)} - V_1^{(L_k)}) = \alpha[(-1)^n V_2^{(L_k)} + (-1)^{n+1}V_1^{(L_k)}];$$

thus,

$$F_{kn+j} = F_j V_{n+1}^{(L_k)} + \alpha\sum_{i=1}^{n}(-1)^{i+1}V_{n+1-i}^{(L_k)},$$

and the identity holds also for $n + 1$, and therefore for all n. □

In particular, for $j = 2$ and $k = 3$, we get the numbers

$$\{F_{3(n-1)+2}\}_{n=1}^{\infty} = \{1, 5, 21, 89, \ldots\},$$

i.e., every third Fibonacci number, which correspond, by (7), to $V_n^{(L_3)} = V_n^{(4)}$. For example, using formula (6) with $m = L_3 = 4$, we get for $n = 3$ (writing in the multinomial coefficients the values $j_0, \ldots, j_4$ from left to right and collecting terms which differ only in the order of the values of j_1, j_2, j_3):

$$\begin{aligned}&\binom{3}{0}\binom{1}{1,0,0,0,1} + \binom{2}{1}\binom{2}{2,0,0,0,0} + 3\binom{1}{1}\binom{1}{0,1,0,0,1} + 3\binom{2}{0}\binom{2}{1,1,0,0,0}\\ &\qquad + 3\binom{1}{0}\binom{2}{0,2,0,0,0} + 3\binom{1}{0}\binom{2}{0,1,1,0,0}\\ &= 1 + 2 + 3 + 6 + 3 + 6 = 21 = V_3^{(4)} = F_8.\end{aligned}$$

4. Concluding Remarks

Combinatorial representations of several recursively defined sequences of integers were generated, using the special properties of the corresponding numeration systems. On the other hand, it may sometimes be desirable to evaluate directly the number of strings satisfying some constraints. The above techniques then suggest to try to define a numeration system accordingly. For example, in Agur and Kerszberg [2] a model of biological processing of genetic information is proposed, in which a binary string symbolizing a DNA sequence is transformed by repeatedly applying some transition function $\mathcal{M}$. For $\mathcal{M}$ being the majority rule, the number of possible final strings, or phenotypes, is evaluated in [1] using the binary numeration system based on the standard Fibonacci numbers. Other transition functions could be studied, and if the resulting phenotypes can be characterized as satisfying some constraints, the corresponding numeration system gives an easy way to evaluate the number of these strings.

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References

1. Z. Agur, A. S. Fraenkel, & S. T. Klein. "The Number of Fixed Points of the Majority Rule." *Discrete Mathematics* *70* (1988):295-302.
2. Z. Agur & M. Kerszberg. "The Emergence of Phenotypic Novelties through Progressive Genetic Change." *Amer. Natur.* *129* (1987):862-75.
3. A. Apostolico & A. S. Fraenkel. "Robust Transmission of Unbounded Strings using Fibonacci Representations." *IEEE Trans. on Inf. Th.*, *IT-33* (1987): 238-45.
4. D. H. Davies. "The CD-ROM Medium." *J. Amer. Soc. Inf. Sc.* *39* (1988):34-42.
5. L. E. Dickson. *History of the Theory of Numbers*, Vol. 1. Washington, D.C.: Carnegie Institution of Washington, 1919.
6. W. Feller. *An Introduction to Probability Theory and Its Applications*, Vol. I. New York: John Wiley & Sons, 1950.
7. A. S. Fraenkel. "Systems of Numeration." *Amer. Math. Monthly* *92* (1985): 105-14.
8. A. S. Fraenkel & S. T. Klein. "Novel Compression of Sparse Bit-Strings—Preliminary Report." *Combinatorial Algorithms on Words*, NATO ASI Series Vol. F12. Berlin: Springer Verlag, 1985, pp. 169-83.
9. A. F. Horadam. "Basic Properties of a Certain Sequence of Numbers." *The Fibonacci Quarterly* *3.2* (1985):169-83.
10. W. H. Kautz. "Fibonacci Codes for Synchronization Control." *IEEE Trans. on Inf. Th.*, *IT-11* (1965):284-92.
11. D. E. Knuth. *The Art of Computer Programming.* Vol. I: *Fundamental Algorithms.* Reading, Mass.: Addison-Wesley, 1973.
12. D. E. Knuth. *The Art of Computer Programming.* Vol. III: *Sorting and Searching.* Reading, Mass.: Addison-Wesley, 1973.
13. E. P. Miles, Jr. "Generalized Fibonacci Numbers and Associated Matrices." *Amer. Math. Monthly* *67* (1960):745-52.
14. A. N. Philippou & D. L. Antzoulakos. "Multivariate Fibonacci Polynomials of Order k and the Multiparameter Negative Binomial Distribution of the Same Order." *Proc. Third International Conference on Fibonacci Numbers and Their Applications* (Pisa 1988). Ed. by G. E. Bergum, A. N. Philippou, and A. F. Horadam. Dordrecht: Kluwer Academic Publishers, 1989.
15. A. N. Philippou & A. A. Muwafi. "Waiting for the k^{th} Consecutive Success and the Fibonacci Sequence of Order k." *The Fibonacci Quarterly* *20.1* (1982):28-32.
16. J. Pihko. "On Fibonacci and Lucas Representations and a Theorem of Lekkerkerker." *The Fibonacci Quarterly* *26.3* (1988):256-61.
17. E. Zeckendorf. "Representation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas." *Bull. Soc. Royale Sci. Liege* *41* (1972):179-82.

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