ON MULTI-SETS

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The n^{th} Fibonacci number F_n and the n^{th} Lucas number L_n are defined by

$$F_1 = 1 = F_2 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3$$

and

$$L_1 = 1,\ L_2 = 3, \text{ and } L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 3,$$

respectively. Thus, the Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ..., and the Lucas sequence is 1, 3, 4, 7, 11, 18, 29, 47, 76, ... Here we have added two adjacent numbers of a sequence and put the result next in the line.

What happens if we put the result in the middle?

Given the initial sets $T_1 = \{1\}$ and $T_2 = \{1, 2\}$, we will get the following increasing sequences of T-sets. These sets are *multi-sets* and the elements are ordered.

$$T_3 = \{1, 3, 2\},\ T_4 = \{1, 4, 3, 5, 2\},\ T_5 = \{1, 5, 4, 7, 3, 8, 5, 7, 2\},$$
$$T_6 = \{1, 6, 5, 9, 4, 11, 7, 10, 3, 11, 8, 13, 5, 12, 7, 9, 2\},$$
$$T_7 = \{1, 7, 6, 11, 5, 14, 9, 13, 4, 15, 11, 18, 7, 17, 10, 13, 3, 14, 11, 19, 8, 21, 13, 18, 5, 17, 12, 19, 7, 16, 9, 11, 2\}, \ldots$$

We show in the following that these multi-sets have some nice and interesting properties.

Proposition 1: Let $|T_n|$ denote the cardinality of the multi-set T_n. Then $|T_n| = 2^{n-2} + 1$ for $n \geq 2$.

Proof: Since $|T_n| = 2^{n-2} + 1$ for $n = 2$, we consider the case $n > 2$ in the following. We obtain T_n from T_{n-1} by inserting a new number in between every pair of consecutive members of T_{n-1} which is their sum. If $|T_{n-1}| = m$, then there are $m - 1$ gaps. In each of these gaps a new number will be inserted to form T_n. Thus,

$$|T_n| = m + m - 1 = 2m - 1 = 2|T_{n-1}| - 1.$$

We have $|T_3| = 3$, $|T_4| = 5$, and $|T_5| = 9$. Looking at these numbers we conjecture that $|T_n| = 2^{n-2} + 1$ for $n > 2$. Our conjecture is true for $n = 3$, 4, and 5. Suppose it is true for $n = k$. Then $|T_k| = 2^{k-2} + 1$. Since $|T_{k+1}| = 2|T_k| - 1$,

$$|T_{k+1}| = 2(2^{k-2} + 1) - 1 = 2^{k-1} + 1 = 2^{(k+1)-2} + 1.$$

Thus, assuming the truth of the conjecture for $n = k$, we proved the truth of the conjecture for $n = k + 1$. Hence, by mathematical induction, our conjecture is true for all integers $n \geq 2$.

Proposition 2: The largest number present in the multi-set T_n is F_{n+1}. Furthermore, T_n contains all the Fibonacci numbers up to F_{n+1}.

Proof: Since we have only F_2 and F_3 in T_2, they will be separated by $F_2 + F_3 = F_4$ in T_3 and we shall have F_2, F_4, F_3 in T_3 with F_4 as the largest number and F_3 as the second largest number. Then, in T_4, F_4 and F_3 will be separated by $F_4 + F_3 = F_5$ and we shall have F_4, F_5, F_3 in T_4 with F_5 as the largest number and F_4, the second largest. By induction, we shall have F_n, F_{n+1} or F_{n+1}, F_n as consecutive members in T_n. Thus, the largest number present in T_n will be F_{n+1}.

Since $T_1 \subset T_2 \subset T_3 \subset \cdots \subset T_n$, T_n contains all of the Fibonacci numbers up to F_{n+1}.

Proposition 3: The multi-set T_n, $n \geq 3$ contains all of the Lucas numbers up to L_{n-1}.

Proof: The multi-set T_3 contains two consecutive members 1 and 3 which are L_1 and L_2. Then T_4 will contain L_1, $L_1 + L_2$, L_2, i.e., L_1, L_3, L_2 as consecutive members. T_5 will contain L_3, $L_3 + L_2$, L_2, i.e., L_3, L_4, L_2 as consecutive members. Thus, by induction, the highest Lucas number present in T_n will be L_{n-1}.

Since $T_1 \subset T_2 \subset \cdots \subset T_n$, T_n will contain all Lucas numbers up to L_{n-1}.

Proposition 4: Any two consecutive members in T_n, $n > 1$, are relatively prime.

Proof: The proposition is true for $n = 2$. Suppose it is true for T_{n-1}, i.e., $(a, b) = 1$ for every pair of consecutive members a and b in T_{n-1}. Let x and y be two consecutive members in T_n. Then, either $x - y$ and y (if $x > y$) or x and $y - x$ (if $y > x$) are consecutive members in T_{n-1}. By assumption, if $x - y$ and y are consecutive, then $(x - y, y) = 1$. Hence, $(x, y) = 1$. Similarly, if $(x, y - x) = 1$, then $(x, y) = 1$. By mathematical induction, the proposition holds for all n.

Proposition 5: The second element of T_n is n and the last but one element of T_n is $2n - 3$.

Proof: The result follows by mathematical induction.

Proposition 6: The numbers 1, 2, 3, 4, and 6 appear once and only once in every T_n, $n \geq 6$ as follows:

(i) The number 1 appears in the first place and $1, n, n - 1$ are consecutive members in T_n.

(ii) The number 2 appears in the $(2^{n-2} + 1)^{\text{th}}$ place and $2n - 5, 2n - 3, 2$ are consecutive members in T_n.

(iii) The number 3 appears in the $(2^{n-3} + 1)^{\text{th}}$ place and $3n - 8, 3, 3n - 7$ are consecutive members in T_n.

(iv) The number 4 appears in the $(2^{n-4} + 1)^{\text{th}}$ place and $4n - 15, 5, 4n - 13$ are consecutive members in T_n.

Proof: Follows by induction.

Theorem 1: For $3 \leq m \leq n$, the multiplicity of m in multi-set T_n is $\frac{1}{2}\phi(m)$, where ϕ is Euler's function.

[$\phi(n)$ is the number of numbers less than n and relatively prime to n. We clearly have $\phi(P) = P - 1$ for a prime P. When n is composite with prime factorization $n = \prod_{i=1}^{r} P_i^{a_i}$, then

$$\phi(n) = n\prod_{i=1}^{r}\left(1 - \frac{1}{P_i}\right).]$$

Proof: To get an m in T_n, a pair (a, b) totalling m should appear in T_{n-1} as consecutive members. Since any two consecutive members in T_{n-1} are relatively prime (Proposition 4), the pair (a, b) must be relatively prime. So we need to know the number of pairs (a, b) with $(a, b) = 1$ and $a + b = m$. Consider $m = a + b$ with $(a, b) = 1$. Then, clearly, $(a, m) = 1 = (b, m)$. Since there are $\phi(m)$ numbers less than m and relatively prime to m, we can chose "a" in $\phi(m)$ ways. Once "a" is chosen, $b = m - a$ is fixed. Since the pairs (a, b) and (b, a) give the same total, we have $\frac{1}{2}\phi(m)$ pairs (a, b) satisfying $(a, b) = 1$ and $(a + b) = m$. Clearly $(1, m - 1)$ is one of the $\frac{1}{2}\phi(m)$ pairs, and this pair appears for the first time (and for the last as well) as consecutive members in T_{m-1}. This pair will yield an m in T_m. Thus, we are guaranteed an appearance of m in T_m.

A natural question is: How many times does m occur in T_m? Since m has $\frac{1}{2}\phi(m)$ pairs (a, b), m can appear at most $\frac{1}{2}\phi(m)$ times in T_m. We prove below that m occurs exactly $\frac{1}{2}\phi(m)$ times in T_m.

Consider a relatively prime pair $(a, m - a)$ with $a < m - a$, $a \neq 1$. Does it belong to T_n for some n? Since $(a, m - a) = 1$, the g.c.d. of "a" and "$m - a$" is 1. Then, by Euclid's g.c.d. algorithm, we have:

$$\begin{array}{llll}
a \,\big)\, m - a \,\big(\, q_1 \\
\quad\;\; aq_1 \\
\quad\;\; \overline{\gamma_1 \,\big)\, a \,\big(\, q_2} \\
\qquad\quad \gamma_1 q_2 \\
\qquad\quad \overline{\gamma_2 \,\big)\, \gamma_1 \,\big(\, q_3} \\
\qquad\qquad \gamma_2 q_3 \\
\qquad\qquad \overline{\gamma_3 \,\big)\, \gamma_2 \,\big(\, q_4} \\
\qquad\qquad\quad \gamma_3 q_4 \\
\qquad\qquad\quad \overline{\gamma_4 \quad \ddots} \\
\qquad\qquad\qquad \gamma_{t-1} \,\big)\, \gamma_{t-2} \,\big(\, q_t \\
\qquad\qquad\qquad\qquad \gamma_{t-1} q_t \\
\qquad\qquad\qquad \overline{1 = \gamma_t \,\big)\, \gamma_{t-1} \,\big(\, \gamma_{t-1}} \\
\qquad\qquad\qquad\qquad\quad \gamma_{t-1} \\
\qquad\qquad\qquad\qquad\quad \overline{\quad 0 \quad}
\end{array}$$

Thus, whenever $(a, m - a) = 1$, we have the last nonzero remainder $\gamma_t = 1$, with the last quotient γ_{t-1}. It is clear that γ_i $(i \neq t) > 1$.

From the algorithm, we obtain:

$$\begin{aligned}
m - a - \gamma_1 &= aq_1 \\
a - \gamma_2 &= \gamma_1 q_2 \\
\gamma_1 - \gamma_3 &= \gamma_2 q_3 \\
&\vdots \\
\gamma_{t-2} - \gamma_t &= \gamma_{t-1} q_t \\
\gamma_{t-1} &= \gamma_t \gamma_{t-1}, \text{ where } \gamma_t = 1 \text{ and } \gamma_i > 1 \text{ for } 1 < i < t.
\end{aligned}$$

Adding, we obtain:

$$m - \gamma_t = aq_1 + \gamma_1 q_2 + \gamma_2 q_3 + \cdots + \gamma_{t-1} q_t + \gamma_{t-1}$$

or $$m - 1 > q_1 + q_2 + q_3 + \cdots + q_t + \gamma_{t-1}.$$

If we start with two consecutive members, a, $m - a$ or $m - a$, a, and proceed backward, we reach the consecutive pair $(1, \gamma_{t-1})$ after $q_1 + q_2 + \cdots + q_t$ steps.

Conversely, if we start with two consecutive members, 1, γ_{t-1}, we reach a consecutive member, a, $m - a$ or $m - a$, a, after $q_t + \cdots + q_3 + q_2 + q_1$ steps.

Since 1, γ_{t-1} are consecutive in the $T_{\gamma_{t-1}}$-set, and nowhere else, the pair $(a, m - a)$ appears as consecutive members in $T_{q_1 + q_2 + \cdots + q_t + \gamma_{t-1}}$.

Since $q_1 + q_2 + \cdots + \gamma_{t-1} < m - 1$, the pair $(a, m - a)$ or $(m - a, a)$ appears as consecutive members in T_i, $i < m - 1$. Thus, every pair $(a, m - a)$ with $(a, m) = 1$, excepting $(1, m - 1)$, appears as consecutive members in some T_i, $i < m - 1$ and the pair $(1, m - 1)$ appears as consecutive in T_m. Hence, for $3 \leq m \leq n$, the multiplicity of m in multi-set T_n is $\frac{1}{2}\phi(m)$. We shall see that, excepting the pair $(1, m - 1)$, other pairs appear in T_i, where $i < [(m + 3)/2]$.

Theorem 2: Every relatively prime pair $(a, m - a)$, $a \neq 1$, $a < m - a$ appears in T_i where $i < [(m + 1)/2]$, we have $i = (m + 1)/2$ in case m is odd.

Proof: We have $m - 1 = aq_1 + \gamma_1 q_2 + \gamma_2 q_3 + \cdots + \gamma_{t-1} q_t + \gamma_{t-1}$, where

$$a > \gamma_1 > \gamma_2 > \gamma_3 > \cdots > \gamma_{t-1} > \gamma_t = 1,$$

and each $q_i \geq 1$. If $\gamma_{t-1} = s$, then $m - 1 > s(q_1 + q_2 + q_3 + \cdots + q_t + 1)$, so

$$\frac{m-1}{s} > q_1 + q_2 + q_3 + \cdots + q_t + 1$$

or

$$\frac{m-1}{s} + s - 1 > q_1 + q_2 + q_3 + \cdots + q_t + s$$

or

$$q_1 + q_2 + q_3 + \cdots + q_t + s \leq \left[\frac{m-1}{s} + s - 1\right],$$

where $[x]$ stands for the greatest integer $\leq x$. The pair $(a, m - a)$ appears in the $(q_1 + q_2 + \cdots + q_t + s)^{\text{th}}$ multi-set. Hence, every pair $(a, m - a)$ of the required type terminating in 1 and s in the g.c.d. algorithm is present as consecutive members in the multi-set T_i, where $i \leq [(m - 1)/s + (s - 1)]$. For $s = 2$,

$$\left[\frac{m-1}{s} + s - 1\right] = \left[\frac{m-1}{2} + 2 - 1\right] = \left[\frac{m+1}{2}\right].$$

If m is odd and $s = 2$, then

$$\left[\frac{m-1}{s} + s - 1\right] = \frac{m+1}{2}.$$

For $s \neq 2$, the inequality

$$\frac{m-1}{s} + s - 1 \leq \frac{m+1}{2}$$

holds

$$\Leftrightarrow 2(m - 1 + s^2 - s) \leq sm + s$$

$$\Leftrightarrow 2s^2 - 3s - 2 \leq m(s - 2) \Leftrightarrow m \geq \frac{2s^2 - 3s - 2}{s - 2},\ s \neq 2,$$

$$\Leftrightarrow m \geq 2s + 1,$$

which is true because $m - a > a > s \Rightarrow m > 2s$, i.e., $m \geq 2s + 1$. Now, the above inequality yields

$$\left[\frac{m-1}{s} + s - 1\right] \leq \left[\frac{m+1}{2}\right].$$

Again, when m is odd, $s = (m - 1)/2$ is an integer and

$$\left[\frac{m-1}{s} + s - 1\right] = \left[2 + \frac{m-1}{2}\right] = \left[\frac{m+1}{2}\right] = \frac{m+1}{2}.$$

Thus, the bound $(m + 1)/2$ is attainable when m is odd and $s = (m - 1)/2$. For example, for $m = 43$, consider the pairs (2, 41) and (21, 22). Both appear in T_{22}. In the first case, $s = 2$; in the second case, $s = 21 = (43 - 1)/2$. Hence every relatively prime pair $(a, m - a)$, $a \neq 1$, $a < m - a$ appears in T_i, where $i \leq [(m + 1)/2]$.

From the above discussion, it is clear that i is much less than $[(m + 1)/2]$ when m is even. For $m = 90$, we have:

> (1, 89) in T_{89}; (7, 83) and (13, 77) in T_{18}; (23, 67) and (43, 47) in T_{15}; (11, 79), (29, 61), (31, 59), and (41, 49) in T_{14}; (17, 73), (19, 71), and (37, 53) in T_{11}. Thus, excepting (1, 89), all other pairs appear as consecutive members in T_i, $i \leq 18$. This is much less than $[(m + 1)/2] = 45$.

We discuss below the appearance of certain special pairs as consecutive members in the multi-sets.

(a) The pair $(1, a)$ is always relatively prime. This pair appears as consecutive members in T_a.
(b) The pair $(a + 1, a)$ is always relatively prime whether a is odd or even. This pair appears as consecutive member in T_{a+1}. For example, 4 and 5 appear as consecutive members in T_5, 9 and 10 in T_{10}.
(c) The pair $(2m - 1, 2)$ is always relatively prime. This pair appears as consecutive members in T_{m+1}, $[m + 1 = (2 + 2m - 1 + 1)/2]$. For example, 5 and 2 in T_4, 13 and 2 in $T_{(2+13+1)/2} = T_8$.
(d) The pair $(a, a + 2)$ is relatively prime if a is odd. We need $1 + (a - 1)/2$ steps to reach this pair if we start from the consecutive members 1, 2. Therefore, the pair $(a, a + 2)$ appears as consecutive members in $T_{[1+(a-1)/2]+2} = T_{(a+5)/2}$. For example, the pair 9 and 11 appear as consecutive members in $T_{(9+5)/2} = T_7$.

We use the above facts in the examples given in Table 1.

TABLE 1

m	Relatively Prime Pairs for a Total m	The Number of the T-Set Where the Pair Appears	The Number of the T-Set Where m Appears Separating This Pair
20	1, 19	19 by (a)	20
	3, 17	5 + 3 by (b) = 8	9
	7, 13	1 + 7 by (b) = 8	9
	9, 11	1 + 6 by (c) = 7	8
33	1, 19	32 by (a)	33
	2, 31	17 by (c)	18
	4, 29	7 + 4 by (a) = 11	12
	5, 28	5 + 1 + 3 by (b) or 5 + 4 by (d) = 9	10
	7, 26	3 + 5 by (d) = 8 or 3 + 1 + 4 by (c) = 8	9
	8, 25	3 + 8 by (a) = 1	12
	10, 23	2 + 3 + 3 by (a) = 8	9
	13, 20	1 + 1 + 7 by (b) or 1 + 1 + 1 + 6 by (a) = 9	10
	14, 19	1 + 2 + 5 by (b) or 1 + 2 + 1 + 4 by (a) = 8	9
	16. 17	1 + 16 by (a) = 17	18
40	1, 39	39 by (a)	40
	3, 37	12 + 3 by (a) = 15	16
	7, 33	4 + 5 by (d) = 9	10
	9, 31	3 + 2 + 4 by (a) = 9	10
	11, 29	2 + 1 + 1 + 4 by (b) = 8	9
	13, 27	2 + 13 by (a) = 15	16
	17. 23	1 + 2 + 6 by (b) or 1 + 2 + 1 + 5 by (a) = 9	10
	19, 21	1 + 9 + 2 by (a) or 1 + 11 by (c0 = 12	13
42	1, 41	41 by (a)	42
	5, 37	7 + 4 by (c) = 11	12
	11, 31	2 + 7 by (a) = 9	10
	13, 29	2 + 4 + 3 by (a) = 9	10
	17, 25	1 + 2 + 8 by (a) = 11	12
	19, 23	1 + 4 + 4 by (b) = 9	10

By Propositions 1 and 2, T_n $(n \geq 2)$ has $2^{n-2} + 1$ members with the highest number F_{n+1}. We have

$$2^{n-2} + 1 = F_{n+1} \text{ for } n = 2, 3, 4$$

and

$$2^{n-2} + 1 > F_{n+1} \text{ for } n > 4.$$

So, for $n > 4$, the multi-set T_n has more elements than the highest number present. Does it contain all numbers 1, 2, 3, 4, ... up to F_{n+1}? We see that T_5 omits 6, T_7 omits 20, and T_8 omits 28, 32, and 33. For 6 we have only one relatively prime pair (1, 5). This pair appears as consecutive members in T_5. So 6 will appear for the first time in T_6. From Table 1, we see that the relatively prime pair (9, 11) for 20 appears as consecutive members in T_7 and other pairs appear later. Therefore, 20 will appear for the first time in T_8. Again, the relatively prime pairs (7, 26), (10, 23), and (14, 19) for 33 appear as consecutive members in T_8 (see Table 1). Therefore, 33 will appear for the first time in T_9 and will appear thrice. Thus, given an integer m, we can always find the T_i where m appears for the first time, and given two integers m and i, we can always say whether m appears in T_i. But, for given i, we do not see how we can tell all the numbers which the multi-set T_i omits unless we construct T_i recursively, and this is a horrible task for large "i."

We conclude this paper with the following problem.

Problem 1: Given a positive integer i, find all numbers m that T_i omits without constructing T_i.

Reference

1. John Turner. Problem H-429. *Fibonacci Quarterly* *27.1* (1989):92.

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