ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-455 *Proposed by T. V. Padma Kumar, Trivandrum, South India*

Characterize, as completely as possible, all "Magic Squares" of the form

a_1	a_2	a_3	a_4
b_1	b_2	b_3	b_4
c_1	c_2	c_3	c_4
d_1	d_2	d_3	d_4

subject to the following constraints:

1. Rows, columns, and diagonals have the same sum
2. $a_1 + a_4 + d_1 + d_4 = b_2 + b_3 + c_2 + c_3 = a_1 + b_1 + a_4 + b_4 = K$
3. $c_1 + d_1 + c_4 + d_4 = a_2 + a_3 + b_2 + b_3 = c_2 + c_3 + d_2 + d_3 = K$
4. $a_1 + a_2 + b_1 + b_2 = c_1 + c_2 + d_1 + d_2 = a_3 + a_4 + b_3 + b_4 = K$
5. $c_3 + c_4 + d_3 + d_4 = c_1 + d_2 + a_3 + b_4 = a_1 + a_2 + d_1 + d_2 = K$
6. $a_3 + a_4 + d_3 + d_4 = b_1 + b_2 + c_1 + c_2 = b_3 + b_4 + c_3 + c_4 = K$
7. $a_2 + a_3 + d_2 + d_3 = b_1 + c_1 + b_4 + c_4 = K$
8. $a_1 + b_1 + c_1 + a_2 + b_2 + a_3 = b_4 + c_3 + c_4 + d_2 + d_3 + d_4 = 3K/2$
9. $b_1 + c_1 + d_1 + c_2 + d_2 + d_3 = a_2 + a_3 + a_4 + b_3 + b_4 + c_4 = 3K/2$
10. $a_2^2 + a_3^2 + d_2^2 + d_3^2 = b_1^2 + c_1^2 + b_4^2 + c_4^2$
11. $c_1^2 + c_2^2 + d_1^2 + d_2^2 = a_3^2 + b_3^2 + a_4^2 + b_4^2$
12. $c_3^2 + c_4^2 + d_3^2 + d_4^2 = a_1^2 + b_1^2 + a_2^2 + b_2^2$
13. $a_1^2 + a_2^2 + a_3^2 + a_4^2 + b_1^2 + b_2^2 + b_3^2 + b_4^2 = M$
14. $c_1^2 + c_2^2 + c_3^2 + c_4^2 + d_1^2 + d_2^2 + d_3^2 + d_4^2 = M$
15. $a_1^2 + b_1^2 + c_1^2 + d_1^2 + a_2^2 + b_2^2 + c_2^2 + d_2^2 = M$
16. $a_3^2 + b_3^2 + c_3^2 + d_3^2 + a_4^2 + b_4^2 + c_4^2 + d_4^2 = M$
17. $a_1 + b_2 + c_3 + d_4 + d_1 + c_2 + b_3 + a_4 = b_1 + c_1 + a_2 + d_2 + a_3 + d_3 + b_4 + c_4$
18. $a_1a_2 + a_3a_4 + b_1b_2 + b_3b_4 = c_1c_2 + c_3c_4 + d_1d_2 + d_3d_4$
19. $a_1b_1 + c_1d_1 + a_2b_2 + c_2d_2 = a_3b_3 + c_3d_3 + a_4b_4 + c_4d_4$

H-456 *Proposed by David Singmaster, Polytechnic of the South Bank, London, England*

Among the Fibonacci numbers, F_n, it is known that: 0, 1, 144 are the only squares; 0, 1, 8 are the only cubes; 0, 1, 3, 21, 55 are the only triangular numbers. (See Luo Ming's article in *The Fibonacci Quarterly* 27.2 [May 1989]: 98-108.)

A. Let $p(m)$ be a polynomial of degree at least 2 in m. Is it true that $p(m) = F_n$ has only finitely many solutions?

B. If we replace F_n by an arbitrary recurrent sequence f_n, we cannot expect a similar result, since f_n can easily be a polynomial in n. Even if we assume the auxiliary equation of our recurrence has no repeated roots, we still cannot expect such a result. For example, if

$$f_n = 6f_{n-1} - 8f_{n-2},\ f_0 = 2,\ f_1 = 6,$$

then

$$f_n = 2^n + 4^n,$$

so every f_n is of the form $p(m) = m^2 + m$. What restriction(s) on f_n is(are) needed to make $f_n = p(m)$ have only finitely many solutions?

Comments: The results quoted have been difficult to establish, so Part A is likely to be quite hard and, hence, Part B may well be extremely hard.

H-457 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let $f(N)$ denote the number of addends in the Zeckendorf decomposition of N. The numerical evidence resulting from a computer experiment suggests the following two conjectures. Can they be proved?

Conjecture 1: For given positive integers k and n, there exists a positive integer n_k (depending on k) such that $f(kF_n)$ has a constant value for $n \geq n_k$.

For example,

$$24F_n = F_{n+6} + F_{n+3} + F_{n+1} + F_{n-4} + F_{n-6} \text{ for } n \geq 8.$$

By inspection, we see that $n_1 = 1$, $n_k = 2$ for $k = 2$ or 3, $n_4 = 4$ and $n_k = 5$ for $5 \leq k \leq 8$.

Conjecture 2: For $k \geq 6$, let us define:

(i) μ, the subscript of the smallest odd-subscripted Lucas number such that $k \leq L_\mu$,

(ii) ν, the subscript of the largest Fibonacci number such that $k > F_\nu + F_{\nu-6}$.

Then, $n_k = \max(\mu, \nu)$.

H-458 *Proposed by Paul Bruckman, Edmonds, WA*

Given an integer $m \geq 0$ and a sequence of natural numbers $a_0, a_1, \ldots, a_m$, form the periodic simple continued fraction (s.c.f.) given by:

(1) $$\theta = [a_0; \overline{a_1, a_2, \ldots, a_2, a_1, 2a_0}].$$

The period is symmetric, except for the final term $2a_0$, and may or may not contain a central term [that is, a_m occurs either once or twice in (1)]. Evaluate θ in terms of nonperiodic s.c.f.'s.

SOLUTIONS

No Doubt

H-437 *Proposed by L. Kuipers, Sierre, Switzerland (Vol. 28, no. 1, February 1990)*

Let x, y, n be Natural numbers, where n is odd. If

(*) $L_n/L_{n+2} < x/y < L_{n+1}/L_{n+3}$, show that $y > (1/5)L_{n+4}$.

Are there fractions, x/y, satisfying (*) for which $y < L_{n+4}$?

Solution by Russell Jay Hendel & Sandra A. Monteferrante, Dowling College, Oakdale, NY

We prove that the rational number with smallest denominator satisfying (*) is F_{n+1}/F_{n+3}. An easy induction then shows that $5F_{n+3} > L_{n+4}$, from which the first assertion readily follows. For $n \geq 1$, $F_{n+3} < L_{n+2} < L_{n+4}$. This answers the second question in the affirmative.

Proof: If $n = 1$, an inspection shows that 1/3 is the rational number with the smallest denominator between 1/4 and 3/7. We therefore assume $n \geq 2$.

First

$$L_n/L_{n+1} = 1/(1 + L_{n-1}/L_n).$$

Hence, the continued fraction expansion of L_n is [0; 1, ..., 1, 3] with $n - 1$ ones. Similarly,

$$L_n/L_{n+2} = 1/(2 + L_{n-1}/L_n)$$

and, therefore, $L_n/L_{n+2} = [0; 2, 1, \ldots, 1, 3]$ with $n - 2$ ones.

Next, let z be a real variable and fix an odd n. Define

$$P_0 = 0,\ Q_0 = 1,\ P_1 = 1,\ Q_1 = 2,\ P_i = F_i,\ Q_i = F_{i+2}\ (\text{for } 2 \leq i \leq n-1),$$

$$P_n(z) = zF_{n-1} + F_{n-2}, \quad \text{and} \quad Q_n(z) = zF_{n+1} + F_n.$$

Define the function $f(z) = P_n(z)/Q_n(z) = [0; 2, 1, \ldots, 1, z]$ with $n - 2$ ones. Then $f(3) = L_n/L_{n+2}$, $f(4/3) = L_{n+1}/L_{n+3}$, and $f(\ )$ maps the open interval, $4/3 < z < 3$ onto the open interval $(L_n/L_{n+2}, L_{n+1}/L_{n+3})$.

It follows that, if $f(z)$ is a rational inside the interval $(f(3), f(4/3))$, then its continued fraction must begin $[0; 2, 1, \ldots, 1, 2, \ldots]$. Clearly, among all such continued fractions, $f(2)$ has the smallest denominator. Since

$$f(2) = P_n(2)/Q_n(2) = F_{n+1}/F_{n+3},$$

the proof is complete.

The above analysis can be generalized to describe other rationals with small denominators. For example: $F_m/F_{m+2} = [0; 2, 1, \ldots, 1, 2]$ with $m - 3$ ones where m is an integer bigger than 3. It follows that F_m/F_{m+2} is always in the open interval $(L_n/L_{n+2}, L_{n+1}/L_{n+3})$, if $m \geq n + 1$. In particular, F_m/F_{m+2} satisfies (*) with $F_{m+2} \leq L_{n+4}$, if $n + 1 \leq m \leq n + 3$.

Also solved by P. Bruckman, R. André-Jeannin, and the proposer.

A Fibonacc-ious Integral

H-438 *Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 28, no. 1, February 1990)*

Define the Fibonacci polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ \text{for } n \ge 2.$$

Show that, for all odd integers $n \ge 3$,

$$\int_{-\infty}^{+\infty} \frac{dx}{F_n(x)} = \frac{\pi}{n}\left(1 + 1/\cos\frac{\pi}{n}\right).$$

Solution by Paul S. Bruckman, Edmonds, WA

As is readily established,

(1) $$F_n(x) = \frac{u^n - v^n}{u - v}, \quad n = 0, 1, 2, \ldots,$$

where

(2) $$u = u(x) = \tfrac{1}{2}(x + \sqrt{x^2 + 4}),\ v = v(x) = \tfrac{1}{2}(x - \sqrt{x^2 + 4}).$$

Let

(3) $$I_n = \int_{-\infty}^{\infty} \frac{dx}{F_n(x)}, \quad \text{for odd } n \ge 3.$$

Note that $F_n(x)$ is an even polynomial (for odd n); hence,

(4) $$I_n = 2\int_0^{\infty} \frac{dx}{F_n(x)}.$$

We may make the substitution: $x = 2\sinh\theta$ in (4); then $u(x) = e^{\theta}$, $v(x) = -e^{-\theta}$, $F_n(x) = \cosh\theta/\cosh n\theta$, and $dx = 2\cosh\theta\, d\theta$. Therefore,

(5) $$I = 4\int_0^{\infty} \cosh^2\theta/\cosh n\theta\, d\theta.$$

Since $n \ge 3$, we see that (5) is well defined; indeed, the integrand may be expanded into a uniformly convergent series. We do so, as follows:

$$\begin{aligned}4\cosh^2\theta/\cosh n\theta &= 2(e^{2\theta} + 2 + e^{-2\theta})/(e^{n\theta} + e^{-n\theta})\\ &= 2e^{(2-n)\theta}\left\{\frac{1 + 2e^{-2\theta} + e^{-4\theta}}{1 + e^{-2n\theta}}\right\}\\ &= 2e^{(2-n)\theta}(1 + 2e^{-2\theta} + e^{-4\theta})\sum_{k=0}^{\infty}(-1)^k e^{-2nk\theta}.\end{aligned}$$

Hence, I_n is equal to:

$$2\sum_{k=0}^{\infty}(-1)^k\int_0^{\infty}[e^{-n\theta(2k+1)+2\theta} + 2e^{-n\theta(2k+1)} + e^{-n\theta(2k+1)-2\theta}]d\theta$$

$$= 2\sum_{k=0}^{\infty}(-1)^k[(n(2k+1) - 2)^{-1} + 2(n(2k+1))^{-1} + (n(2k+1) + 2)^{-1}],$$

or, after some simplification:

(6) $$I_n = \frac{4}{n}\sum_{k=0}^{\infty}(-1)^k/(2k+1) + 4n\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)}{(2k+1)^2 n^2 - 4}.$$

The first series in (6) is the well-known Leibnitz series for $\frac{1}{4}\pi$.

The second series in (6) may be evaluated from the Mittag-Leffler formula (see [1]):

(7) $$\pi\sec\pi z = \sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)}{(k+\frac{1}{2})^2 - z^2}, \text{ provided } (z - \tfrac{1}{2}) \notin \mathbb{Z}.$$

Setting $z = 1/n$ in (7), we obtain:

$$\pi \sec \pi/n = 4n^2 \sum_{k=0}^{\infty} (-1)^k (2k + 1)[(2k + 1)^2 n^2 - 4]^{-1}.$$

Comparing this with the second series in (6) yields the desired result:

$$I_n = \pi/n(1 + \sec \pi/n).$$

NOTE: By similar methods, we may prove the following result:

$$\int_{-\infty}^{\infty} x\, dx/F_n(x) = \pi/n(\tan \pi/2n + \tan 3\pi/2n), \text{ if } n \geq 4 \text{ is even.}$$

Reference

1. Louis L. Pennisi. *Elements of Complex Variables*, 2nd ed. Urbana: University of Illinois, 1976, p. 336.

Also solved by P. Byrd, R. André-Jeannin, Y. H. Kwong, N. A. Volodin, and the proposer.

Another Lucas Congruence

H-439 *Proposed by Richard André-Jeannin, ENIS BP W, Tunisia (Vol. 28, no. 1, February 1990)*

Let p be a prime number ($p \neq 2$) and m a Natural number. Show that

$$L_{2p^m} + L_{4p^m} + \cdots + L_{(p-1)p^m} \equiv 0 \pmod{p^{m+1}}.$$

Solution by the proposer

From the formula:

$$a^p + b^p = (a + b)\left[(-1)^{\frac{p-1}{2}}(ab)^{\frac{p-1}{2}} + \sum_{k=1}^{\frac{p-1}{2}} (-1)^{k-1}(ab)^{k-1}(a^{p-2k+1} + b^{p-2k+1})\right],$$

we get, when taking $a = \alpha^{p^m}$, $b = \beta^{p^m}$,

$$L_{p^{m+1}} = L_{p^m}[1 + L_{(p-1)p^m} + L_{(p-3)p^m} + \cdots + L_{2p^m}],$$

hence:

(1) $\qquad L_{p^{m+1}} - L_{p^m} = L_{p^m}[L_{(p-1)p^m} + \cdots + L_{2p^m}].$

But it is known (see Jarden, *Recurring Sequences*, p. 111) that:

(2) $\qquad L_{p^{m+1}} \equiv L_{p^m} \pmod{p^{m+1}}$

and thus (1) becomes:

(3) $\qquad 0 \equiv L_{p^m}[L_{(p-1)p^m} + \cdots + L_{2p^m}] \pmod{p^{m+1}}.$

Now we have: $\gcd(p, L_{p^m}) = 1$ [since, by (2): $L_{p^m} \equiv 1 \pmod{p}$]. Thus, (3) shows that

$$L_{(p-1)p^m} + \cdots + L_{2p^m} \equiv 0 \pmod{p^{m+1}}.$$

Also solved by P. Bruckman and G. Wulczyn.

A Square Product

H-440 *Proposed by T. V. Padma Kumar, Trivandrum, South India (Vol. 28, no. 2, May 1990)*

NOTE: This is the same as H-448.

If $a_1, a_2, \ldots, a_m, n$ are positive integers such that $n > a_1, a_2, \ldots, a_m$ and $\emptyset(n) = m$ and a_i is relatively prime to n for $i = 1, 2, 3, \ldots, m$, prove

$$\left(\prod_{i=1}^{m} a_i\right)^2 \equiv 1 \pmod{n}.$$

Solution by Sahib Singh, Clarion University of Pennsylvania, Clarion, PA

Consider the ring $(Z_n, +_n, \cdot_n)$ with $Z_n = \{0, 1, 2, \ldots, (n-1)\}$, where the operations are addition modulo n and multiplication modulo n, respectively. Under this hypothesis, the given members $a_1, a_2, \ldots, a_m$ are precisely the members of the multiplicative group of units of this ring. These m units can be partitioned into two classes. The first class consists of those members a_i (as well as a_t) such that

$$a_i a_t \equiv 1 \pmod{n}, \text{ where } i \neq t;\ 1 \leq i,\ t \leq m.$$

The second class contains the remaining members a_j that satisfy $a_j^2 \equiv 1 \pmod{n}$.

Without loss of generality, we can name the members of the first class as $a_1, a_2, \ldots, a_k$ and the members of the second class as $a_{k+1}, a_{k+2}, \ldots, a_m$. (Note that it is possible that the first class is empty, so that $k = 0$: this can be verified when $n = 8$.)

Consequently,

$$\prod_{i=1}^{m} a_i = \left(\prod_{i=1}^{k} a_i\right)(a_{k+1} \cdot a_{k+2} \cdot \cdots \cdot a_m).$$

Since $\prod_{i=1}^{k} a_i \equiv 1 \pmod{n}$, we conclude that:

$$\left(\prod_{i=1}^{m} a_i\right)^2 = \left(\prod_{i=1}^{k} a_i\right)^2 (a_{k+1} \cdot a_{k+2} \cdot \cdots \cdot a_m)^2 \equiv 1 \pmod{n}.$$

Also solved by P. Bruckman, B. Prielipp, and L. Somer.

Editorial Notes:

1. Lawrence Somer's name was inadvertently omitted as a solver of H-424.

2. A number of readers pointed out that H-451 is the same as B-643.

3. Paul Bruckman's name was inadvertently omitted as a solver of H-434. He mentioned that line one of the solution should read "$[c_1 r_1^n + \frac{1}{2}]$" and that the value reported for the approximation of c_1 should be 1.22041 not 1.22144.

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