SEQUENCES OF INTEGERS SATISFYING RECURRENCE RELATIONS

Richard André-Jeannin
Ecole Nationale d'Ingénieurs de Sfax, Tunisia
(Submitted July 1989)

Let us consider the recurrence relation

(1) $$n^3u_n - (34n^3 - 51n^2 + 27n - 5)u_{n-1} + (n - 1)^3u_{n-2} = 0.$$

Apery has proved that for $(u_0, u_1) = (1, 5)$ all of the u_n's are integers, and it is proved in [1] and [2] that, if all the numbers of a sequence satisfying (1) are integers, then $(u_0, u_1) = \lambda(1, 5)$, where λ is an integer. We give here a generalization of this result, with a simple proof, and applications to Apery's numbers as well as to the recurrence relation

(2) $$L_{n-1}F_nu_n - 5F_nF_{n-1}F_{2n-1}u_{n-1} - F_{n-1}L_nu_{n-2} = 0,$$

where F_n, L_n are the usual Fibonacci and Lucas numbers.

1. The Main Result

Let $\{a_n\}$, $\{b_n\}$ be two sequences of rational numbers with $\{u_n\}$ the sequence defined by (u_0, u_1) and the recurrence relation

(3) $$u_n = a_nu_{n-1} + b_nu_{n-2}, \; n \geq 2.$$

We then have two results.

Theorem 1: Suppose that

(4) a) For all integers $n \geq 2$, $b_n \neq 0$.

(5) b) There exists a real number P such that $\lim_{n \to \infty} \prod_{k=2}^{n} |b_k| = P$.

Then the recurrence relation (3) has two linearly independent integer solutions only if $|b_n| = 1$ for all large n.

THeorem 2: Suppose that

(6) a) For all $n \geq 2$, $b_n \neq 0$ and $|b_n| = 1$ for all large n.

(7) b) For all $n \geq 2$, $a_n \neq 0$ and $\lim_{n \to \infty}|a_n| = a$.

Then relation (3) has two linearly independent integer solutions only if $a_n = a$ for all large n, where a is an integer different from zero.

Remark: Recall that two sequences $\{p_n\}$ and $\{q_n\}$ are linearly dependent if two numbers (λ, μ) exist (not both zero) such that, for all n,

$$\lambda p_n + \mu q_n = 0.$$

In the other case, the sequences are linearly independent. It is easy to prove that $\{p_n\}$ and $\{q_n\}$, when satisfying (3), are linearly dependent if and only if

(8) $$p_0q_1 - p_1q_0 = 0.$$

2. Proof of Theorem 1

Let us suppose that $\{p_n\}$ and $\{q_n\}$ are two independent integer solutions of (3) and define the sequence Δ_n by

(9) $$\Delta_n = p_{n-1}q_n - p_nq_{n-1}, \; n \geq 1.$$

It is easily proved that

(10) $\Delta_n = -b_n \Delta_{n-1}, \ n \geq 2.$

Hence,

(11) $\Delta_n = (-1)^{n-1} b_2 \ \dots \ b_n \Delta_1, \ n \geq 2.$

By the Remark above, $\Delta_1 = p_0 q_1 - p_1 q_0 \neq 0$, and by (5) we have

(12) $\lim_{n \to \infty} |\Delta_n| = |\Delta_1| P;$

thus, the sequence of integers $|\Delta_n|$ converges and we deduce from (12) that

(13) $|\Delta_n| = |\Delta_1| P$, for all large n.

By (11) we have $\Delta_n \neq 0$ for all n (since $b_n \neq 0$ and $\Delta_1 \neq 0$). Hence, (13) shows that $P \neq 0$. By (10) we have

$$1 = \frac{|\Delta_n|}{|\Delta_{n-1}|} = |b_n|, \text{ for all large } n.$$

This concludes the proof of Theorem 1.

3. Proof of Theorem 2

Suppose that $\{p_n\}$ and $\{q_n\}$ are two independent integer solutions of (3) and define the sequence D_n of integers by

$$D_n = p_{n-2} q_n - p_n q_{n-2}, \ n \geq 2.$$

It is obvious that

(14) $D_n = a_n \Delta_{n-1}, \ n \geq 2.$

However, by (6) we have, for n large, since $|b_n| = 1$,

$$|\Delta_n| = |\Delta_1| P \neq 0.$$

Hence,

(15) $|D_n| = |a_n| |\Delta_1| P \neq 0$, for all large n,

and by (7),

$$\lim_{n \to \infty} |D_n| = a |\Delta_1| P.$$

Thus, for all large n,

(16) $|D_n| = a |\Delta_1| P.$

Note that $a \neq 0$, since $D_n \neq 0$, and that a is a rational number by (16). Comparison of (15) and (16) shows that

$$|a_n| = a, \text{ for all large } n.$$

Let us now write $a = p/q$, where p and q are relatively prime integers. Without loss of generality, we can assume that

(17) $u_n = \pm \frac{p}{q} u_{n-1} \pm u_{n-2}$, for $n \geq 2$.

Consider the solution $\{v_n\}$ of (17) defined by the initial values (0, 1). Note that $\Delta_1 v_n$ is an integer, namely,

$$\Delta_1 v_n = -q_0 p_n + p_0 q_n.$$

The relation

$$\Delta_1 v_n = \pm \frac{p}{q} \Delta_1 v_{n-1} \pm \Delta_1 v_{n-2}$$

shows that

$$q|\Delta_1 v_{n-1}, \text{ for } n \geq 2.$$

By mathematical induction, it is easy to prove that for all integers $m \geq 1$ and $n \geq 1$, $q^m|\Delta_1 v_n$. Therefore, $q = 1$, and a is an integer.

4. Application

Suppose that $|b_n| = C_{n-1}/C_n$, with $C_n \neq 0$ for all n, $C_n \neq C_{n-1}$, and

$$\lim_{n\to\infty} C_n = C.$$

We can then write

$$\prod_{k=2}^{n} |b_k| = \frac{C_1}{C_n}, \text{ so that } P = \frac{C_1}{C}.$$

By Theorem 1, the sequence (3) cannot have two linearly independent solutions, since $|b_n| \neq 1$.

This result can be applied to (1) with $C_n = n^3$, and also to the recurrences

$$nu_n - (2m + 1)(2n - 1)u_{n-1} + (n - 1)u_{n-2} = 0, \tag{18}$$

and

$$n^2u_n - (11n^2 - 11n + 3)u_{n-1} - (n - 1)^2u_{n-2} = 0, \tag{19}$$

with $C_n = n$ in (18), $C_n = n^2$ in (19). Note that (18) and (19) admit integer solutions defined by the initial values $(1, 2m + 1)$ [resp. $(1, 3)$]. The integer solution of (18) is simply $u_n = P_n(-m)$, where

$$P_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}[x^n(1 - x)^n] = \prod_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k x^k$$

is the n^{th} Legendre polynomial over [0, 1] (see [3] for another proof). Equations (1) and (19) appear in Apery's proof of the irrationality of $\zeta(3)$ and $\zeta(2)$.

Now, let us consider recurrence (2), in which we have

$$b_n = \frac{F_{n-1}L_n}{L_{n-1}F_n}.$$

Then

$$\prod_{k=2}^{n} b_k = \frac{L_n}{F_n} \quad \text{and} \quad P = \lim_{n\to\infty}\frac{L_n}{F_n} = \sqrt{5}.$$

By Theorem 1, the sequence (2) cannot have two linearly independent integer solutions. it will be shown below (and in [4]) that the solution $\{q_n\}$ defined by the initial values (1, 0) is an integer sequence. On the other hand, the solution $\{p_n\}$ defined by the initial values (0, 1) cannot be an integer sequence. Let us write the first few values of these two sequences in order to see this. They are:

n	0	1	2	3	4	5 ...
p_n	0	1	10	84	$\frac{8225}{3}$	$\frac{999146}{5}$ ...
q_n	1	0	3	25	816	59475 ...

It can also be shown that

$$\lim_{n\to\infty}\frac{p_n}{q_n} = \sum_{k=1}^{\infty}\frac{1}{F_k} = 3.35988566624\ldots .$$

Notice how quickly p_n/q_n converges. We have

$$\frac{p_4}{q_4} = 3.3598856\ldots \quad \text{and} \quad \frac{p_5}{q_5} = 3.35988566624\ldots .$$

One can deduce from this that $\sum_{k=1}^{\infty}(1/F_k)$ is irrational (see [4]).

5. Generalization

Consider the recurring sequence defined by $u_0, \ldots, u_{r-1}$ and

$$u_n = a_n^1 u_{n-1} + a_n^2 u_{n-2} + \cdots + a_n^r u_{n-r}, \; n \geq r, \tag{20}$$

where r is a strictly positive integer, and where $\{a_n^1\}, \ldots, \{a_n^r\}$ are sequences of rational numbers. By analogy with Theorem 1, we have the following result.

Theorem 1': Suppose that

(a) For all $n \geq r$, $a_n^r \neq 0$.

(b) There exists a number P such that $\lim_{n\to\infty} \prod_{k=r}^{n} |a_k^r| = P$.

Then (20) has r linearly independent integer solutions only if $|a_n^r| = 1$ for all large n.

Proof: Suppose that $\{p_n^1\}, \ldots, \{p_n^r\}$ are r linearly independent integer sequence solutions of (20) and define the sequence Δ_n of integers by $r \times r$ determinant

$$\Delta_n = \left| p_{n-r+j} \right|_{\substack{1 \leq i \leq r \\ 1 \leq j \leq r}}, \; n \geq r - 1.$$

It is easily proved that $\Delta_n = (-1)^{r-1} a_n \Delta_{n-1}$. Hence,

$$|\Delta_n| = |\Delta_{r-1}| \prod_{k=r}^{n} |a_k^r|, \; n \geq r.$$

We have $\Delta_{r-1} \neq 0$, since the $\{p_n^i\}$'s are independent, and the end of the proof is as in Theorem 1.

The reader can also find a theorem analogous to Theorem 2.

References

1. Y. Mimura. "Congruence Properties of Apery Numbers." *J. Number Theory* *16* (1983):138-46.
2. D. K. Chang. "A Note on Apery Numbers." *Fibonacci Quarterly* *22.2* (1984): 178-80.
3. D. K. Chang. "Recurrence Relations and Integer Sequences." *Utilitas Mathematica* *32* (1987):95-99.
4. R. André-Jeannin. "Irrationalité de la somme des inverses de certaines suites récurrentes." *C. R. Acad. Sci. Paris* *t. 308, série I* (1989):539-41.

*****