THE ZECKENDORF REPRESENTATION AND THE GOLDEN SEQUENCE

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Preamble

In what follows, we have

The Golden section: $\tau = \dfrac{\sqrt{5} - 1}{2} = 0.618\ldots$.

Fibonacci numbers: $F_0 = 0,\ F_1 = 1,\ F_i = F_{i-1} + F_{i-2},\ i \geq 2.$

The Zeckendorf representation of a number is simply the representation of that number as the sum of distinct Fibonacci numbers. If the number of terms of this sum is minimized, that representation is unique, as also is the representation when the number of terms is maximized. (See Brown [1] and [2].)

A *general* Zeckendorf representation will be written as

$$\sum_{j=1}^{h} F_{k_j}, \quad \text{where } k_1 > k_2 > \cdots > k_h \geq 2.$$

Thus, 16 can be represented as

$$F_7 + F_4,\ F_6 + F_5 + F_4,\ F_7 + F_3 + F_2, \text{ and } F_6 + F_5 + F_3 + F_2.$$

The first is the unique minimal representation; the last is the unique maximal representation. The others show that representations of any intermediate length need not be unique.

It is easy to show that only numbers of the form $F_n - 1$ have a unique Zeckendorf representation (i.e., one that is maximal and minimal).

From here on, we will refer to the minimal Zeckendorf representation and the maximal Zeckendorf representation as the *minimal* and *maximal*.

We define

Beta-sequence: $\{\beta_j\},\ j = 1, 2, 3, \ldots,\ \beta_j = [(j + 1)\tau] - [j\tau]$.
This takes on only the values zero or unity.

Golden sequence: Any sequence such as *abaababa*... which is obtained from the Beta-sequence $\beta_1, \beta_2, \beta_3, \ldots$, where "$b$" corresponds to a zero and "a" corresponds to a unit.

We will prove that the final term of each maximal representation is either F_2 or F_3 and show the pattern associated with the final terms in the representations of 1, 2, 3, 4, 5, 6, ..., namely: $F_2, F_3, F_2, F_2, F_3, F_2, \ldots$ is a Golden sequence with the term F_2 corresponding to a unit and the term F_3 corresponding to a zero.

More specifically, we will show that the last term in the maximal representation of the number n is $F_{3-\beta_n} = 2 - \beta_n$.

We note a similar result for the "modified" Zeckendorf representation which may include F_1 as well as F_2.

Main Results

Theorem 1: The maximal ends with F_2 or F_3.

Proof: We note that F_3 cannot be replaced by $F_2 + F_1$ in a Zeckendorf expansion as $F_2 = F_1$. If F_k with $k > 3$ is the smallest term in an expansion of a number

n, then F_k can be replaced by $F_{k-1} + F_{k-2}$ and so the expansion is not maximal. Thus, if an expansion is maximal, it must end in F_2 or F_3.

Lemma 1: $[(j + F_i)\tau] = F_{i-1} + [j\tau]$ if $i \geq 2$ and $0 < j < F_{i+1}$.

Proof: Fraenkel, Muchkin, and Tassa proved in [3] that if θ is irrational, $0 < j < q_i$ and p_i/q_i is the i^{th} convergent to θ in the elementary theory of continued fractions, then

$$[(j + q_{i-1})\theta] = p_{i-1} + [j\theta], \; i \geq 1.$$

As F_{i-1}/F_i is a convergent to τ, our result follows.

Lemma 2: If $\sum_{j=1}^{h} F_{k_j}$ is a Zeckendorf expansion, then $\sum_{j=2}^{h} F_{k_j} < F_{k_1+1} - 1$.

Proof: $\sum_{j=2}^{h} F_{k_j} \leq F_{k_1-1} + F_{k_1-2} + \cdots + F_2 = F_{k_1+1} - 2$, since $\sum_{i=1}^{n} F_i = F_{n+2} - 1$.

The result is now obvious.

Lemma 3: If j has a Zeckendorf expansion $\sum_{i=1}^{h} F_{k_i}$, then

(a) $[j\tau] = F_{k_1-1} + F_{k_2-1} + \cdots + F_{k_{h-1}-1} + [\tau F_{k_h}]$

(b) $[(j + 1)\tau] = F_{k_1-1} + F_{k_2-1} + \cdots + F_{k_{h-1}-1} + F_{k_h-1}$.

Proof:

(a) Let $m = \sum_{i=2}^{h} F_{k_i}$, then by Lemma 2, $m < F_{k_1+1}$ and so by Lemma 1,

$$[j\tau] = [(F_{k_1} + m)\tau] = F_{k_1-1} + [m\tau].$$

Similarly, if $n = \sum_{i=3}^{h} F_{k_i}$, $[m\tau] = F_{k_2-1} + [n\tau]$, so eventually

$$[j\tau] = F_{k_1-1} + \cdots + F_{k_{h-1}-1} + [\tau F_{k_h}].$$

(b) As in (a) (this time with $m + 1 < F_{k_1+1}$),

$$\begin{aligned}[(j + 1)\tau] &= [(F_{k_1} + \cdots + (F_{k_h} + 1))\tau] \\ &= F_{k_1-1} + \cdots + F_{k_{h-1}-1} + [(F_{k_h} + 1)\tau] \\ &= F_{k_1-1} + \cdots + F_{k_{h-1}-1} + F_{k_h-1} \text{ by Lemma 1.}\end{aligned}$$

Lemma 4: If j has a maximal $\sum_{i=1}^{h} F_{k_i}$, then

(a) $[j\tau] = F_{k_1-1} + \cdots + F_{k_{h-1}-1} + F_{k_h} - 1$.

(b) $\beta_j = 2 - F_{k_h}$.

Proof:

(a) If $k_h = 2$, then $[\tau F_{k_h}] = 0 = F_{k_h} - 1$.

If $k_h = 3$, then $[\tau F_{k_h}] = 1 = F_{k_h} - 1$,

so the result follows by Lemma 3(a).

(b) By Lemmas 4(a) and 3(b),

$$\beta_j = [(j + 1)\tau] - [j\tau] = F_{k_h-1} - F_{k_h} + 1 = 2 - F_{k_h},$$

as $k_h = 2$ or 3 and so $F_{k_h-1} = 1$.

Theorem 2: The last term in the maximal for j is $F_{3-\beta_j} = 2 - \beta_j$.

Proof: By Lemma 4(b), if F_{k_h} is the last term in a maximal for j, then

$$\beta_j = 2 - F_{k_h}.$$

If $k_h = 3$, then $\beta_j = 0$ and $F_{3-\beta_j} = F_3 = 2 - \beta_j$.

If $k_h = 2$, then $\beta_j = 1$ and $F_{3-\beta_j} = F_2 = 2 - \beta_j$.

We now see that the last term of the maximal for any integer j is either 1 or 2. It also follows immediately that the sequence of the last terms for the maximals for 1, 2, 3, 4, ... form a Golden sequence 1211212112..., where a unit is unchanged but a zero is replaced by 2.

Suppose we form the modified maximal from the maximal by forcing the last term to be unity; that is, the last two terms are $F_3 + F_2$, $F_3 + F_1$, or $F_2 + F_1$. Then it follows easily from the above that the second last terms of the modified maximals for 2, 3, 4, ... correspond to the same golden pattern as the last terms in the maximals for 1, 2, 3, ...

References

1. J. L. Brown. "Zeckendorf's Theorem and Some Applications." *Fibonacci Quarterly* *2.2* (1964):163-68.
2. J. L. Brown. "Z New Characterization of the Fibonacci Numbers." *Fibonacci Quarterly* *3.1* (1965):1-8.
3. A. S. Fraenkel, M. Mushkin, & U. Tassa. "Determination of $[n\theta]$ by Its Sequence of Differences." *Can. Math. Bull.* *21* (1978):441-46.

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