A NOTE ON EULER'S NUMBERS

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Recently, Y. Imai, Y. Seto, S. Tanaka, and H. Yutani [1] defined the coefficients $Z(m, r)$ by

$$(1) \qquad Z(m, r) = \sum_{k=1}^{r} (-1)^{r+k} \binom{m+1}{r-k} k^m \quad (m \geq 1,\ r = 1, \ldots, m),$$

$$Z(m, r) = 0 \quad (m \leq 0 \text{ or } r \leq 0 \text{ or } m < r),$$

and proved that

$$x^m = \sum_{r=1}^{m} \left(\frac{Z(m, r)}{m!} \prod_{i=1}^{m} (x + i - r) \right) \quad (x, m \in N),$$

$$Z(m, r) = Z(m, m + 1 - r),$$

$$(2) \qquad \sum_{r=1}^{m} Z(m, r) = m! \quad (m \geq 1,\ r = 1, \ldots, m),$$

$$Z(m + 1, r) = (m - r + 2)Z(m, r - 1) + rZ(m, r).$$

In this short note we will show that the coefficients $Z(m, r)$ are just Euler's numbers $A_{m,r}$ introduced in 1755 by

$$A_{m,r} = \sum_{k=0}^{r-1} (-1)^k \binom{m+1}{k} (r-k)^m.$$

Indeed, using the substitution $j = r - k$, from (1) follows

$$\sum_{k=1}^{r} (-1)^{r+k} \binom{m+1}{r-k} k^m = \sum_{j=0}^{r-1} (-1)^j \binom{m+1}{j} (r-j)^m,$$

i.e., that $Z(m, r) = A_{m,r}$.

In [1] the authors mentioned that it would be interesting to find a connection between the coefficients $Z(m, r)$ and Stirling's numbers of the second kind $S(n, k)$. Since $Z(m, r) = A_{m,r}$, we have the following relations (see, e.g., [2], [3])

$$Z(m, r) = \sum_{k=0}^{m-r} (-1)^k \binom{r+k-1}{k-1} (m-k-r+1)!S(m, m-r-k+1),$$

$$Z(m, r) = \sum_{k=1}^{m} (-1)^{m-r+k-1} \binom{m-k}{k-1} k!S(m, k),$$

$$k!S(m, k) = \sum_{r=1}^{m} Z(m, r) \binom{m-r}{m-k}.$$

If we take $m = k$ in the last equality, we obtain

$$m!S(m, m) = \sum_{r=1}^{m} Z(m, r),$$

which is equivalent to (2), because $S(m, m) = 1$. This is Lemma 2 from [1].

References

1. Y. Imai, Y. Seto, S. Tanaka, & H. Yutani. "An Expansion of x^m and Its Coefficients." *Fibonacci Quarterly* *26.1* (1988):33-39.
2. L. Toscano. "Sulla iterazione dell'operatore xD." *Rendiconti di Mathematica e delle sue applicazioni* *(5) 8* (1949):337-50.
3. L. Toscano. "Su una relazione di ricorrenza triangolare." *Ibid.* *(5) 8* (1950):247-54.

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