ENTRY POINT RECIPROCITY OF CHARACTERISTIC CONJUGATE GENERALIZED FIBONACCI SEQUENCES

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Introduction

Given a pair of integers, A, B, such that $(A, B) = 1$ and $0 < A < \frac{1}{2}B$, we define a generalized Fibonacci sequence as follows:

$$G_0 = B - A,\ G_1 = A,\ G_n = G_{n-1} + G_{n-2} \quad \text{for } n \geq 2.$$

Terms with negative indices can also be defined by:

$$G_{-n} = G_{2-n} - G_{1-n} \quad \text{for } n \geq 1.$$

We say that

$$|G_1^2 - G_0G_2| = |A^2 + AB - B^2|$$

is the *characteristic* of $\{G_n\}$. In addition, we define a *conjugate sequence* $\{H_n\}$ by:

$$H_0 = B - A,\ H_1 = B - 2A,\ H_n = H_{n-1} + H_{n-2} \quad \text{for } n \geq 2.$$

It is easily seen that:

1. $G_n > 0$ and $H_n > 0$ for all $n \geq 0$;
2. $H_n = (-1)^n G_{-n} = |G_{-n}|$;
3. $\{G_n\}$ and $\{H_n\}$ have the same characteristic;
4. $\{G_n\}$ and $\{H_n\}$ are distinct unless $A = 1$, $B = 3$, in which case $G_n = H_n = L_n$ (the n^{th} Lucas number; see [1]).

Let $\{T_n\} = \{G_n\}$ or $\{H_n\}$. If M is any positive integer, we say M enters $\{T_n\}$ if there exists $K > 0$ such that $M | T_K$. The least such K will be called the *entry point* of M in $\{T_n\}$, and denoted $T(M)$. The entry point of M in the original Fibonacci sequence $\{F_n\}$ (which is guaranteed to exist) is denoted $Z(M)$. The entry point of M (if it exists) in $\{L_n\}$, $\{G_n\}$, $\{H_n\}$ will be denoted $L(M)$, $G(M)$, $H(M)$, respectively.

In this paper we prove the following theorems.

Theorem 1: If $M | G_0$, then M enters $\{G_n\}$ and $\{H_n\}$, and $G(M) = H(M) = Z(M)$.

Theorem 2: If $M \nmid G_0$ but M enters $\{G_n\}$, then M also enters $\{H_n\}$, and $G(M) + H(M) = Z(M)$.

Theorem 2 may be considered an entry point reciprocity law. We will make use of the following identities.

(1) $T_{m+n} = F_{m-1}T_n + F_m T_{n+1}$

(2) $G_n = F_{n-2}A + F_{n-1}B$

(3) $H_n = -F_{n+2}A + F_{n+1}B$

(4) $(T_n, T_{n+1}) = (F_n, F_{n+1}) = 1$

(5) $F_{-n} = (-1)^{n-1}F_n$

(6) $F_{m+n} = F_{m-1}F_n + F_m F_{n+1}$

The Main Results

Proof of Theorem 1: Since $G_0 = H_0 = B - A$, and $(G_0, G_1) = (H_0, H_1) = 1$, it suffices to show that, if $\{T_n\}$ is a sequence such that $M|T_0$ and $(T_0, T_1) = 1$, then M enters $\{T_n\}$ and $T(M) = Z(M)$. (1) implies $T_K = F_{K-1}T_0 + F_K T_1$; therefore, hypothesis implies $T_K \equiv F_K T_1 \pmod{M}$, so that

$$T_{Z(M)} \equiv F_{Z(M)}T_1 \equiv 0 \pmod{M}.$$

Thus, M enters $\{T_n\}$ and $T(M) \leq Z(M)$. Also

$$F_{T(M)}T_1 \equiv T_{T(M)} \equiv 0 \pmod{M}.$$

But $(T_0, T_1) = 1$, so $(M, T_1) = 1$. Therefore, $F_{T(M)} \equiv 0 \pmod{M}$. This implies $Z(M) \leq T(M)$, so $T(M) = Z(M)$.

Lemma 1: Let $\{T_n\} = \{G_n\}$ or $\{H_n\}$. If X is an integer such that $0 < X < Z(M)$ and $T_X \equiv 0 \pmod{M}$, then $X = T(M)$.

Proof: Hypothesis implies $T(M) \leq X$. Suppose $T(M) = Y < X$. (1) implies

$$T_X = T_{(X-Y)+Y} = F_{X-Y-1}T_Y + F_{X-Y}T_{Y+1}.$$

Thus,

$$T_X \equiv F_{X-Y-1}T_Y + F_{X-Y}T_{Y+1} \pmod{M}.$$

But hypothesis implies $T_X \equiv T_Y \equiv 0 \pmod{M}$, so $F_{X-Y}T_{Y+1} \equiv 0 \pmod{M}$. Hypothesis and (4) imply $(T_Y, T_{Y+1}) = 1$, so that $(M, T_{Y+1}) = 1$. Therefore, $F_{X-Y} \equiv 0 \pmod{M}$. But $0 < X - Y < X < Z(M)$, which contradicts the definition of $Z(M)$. Hence, $T(M) = X$.

Proof of Theorem 2: Let $n = G(M)$. Hypothesis and (2) imply $F_{n-2}A + F_{n-1}B \equiv 0 \pmod{M}$. (3) implies

$$H_{Z(M)-n} = -F_{Z(M)+2-n}A + F_{Z(M)+1-n}B.$$

Now (6) implies

$$F_{Z(M)+2-n} = F_{1-n}F_{Z(M)} + F_{2-n}F_{Z(M)+1} \equiv F_{2-n}F_{Z(M)+1} \equiv (-1)^{n-1}F_{n-2}F_{Z(M)+1} \pmod{M};$$

$$F_{Z(M)+1-n} = F_{-n}F_{Z(M)} + F_{1-n}F_{Z(M)+1} \equiv F_{1-n}F_{Z(M)+1} \equiv (-1)^n F_{n-1}F_{Z(M)+1} \pmod{M}.$$

[The last steps involved use of (5).] Therefore,

$$\begin{aligned} H_{Z(M)-n} &\equiv (-1)^n F_{n-2}F_{Z(M)+1}A + (-1)^n F_{n-1}F_{Z(M)+1}B \\ &\equiv (-1)^n F_{Z(M)+1}(F_{n-2}A + F_{n-1}B) \equiv 0 \pmod{M}. \end{aligned}$$

Thus, by Lemma 1,

$$H(M) = Z(M) - n = Z(M) - G(M).$$

Corollary 1: For $\{T_n\}$, if $T(M)$ exists, then $T(M) \leq Z(M)$; if $T(M) = Z(M)$, then $M|T_0$.

This follows from Theorems 1 and 2.

Corollary 2: If M enters $\{L_n\}$ and $M > 2$, then $L(M) = \frac{1}{2}Z(M)$; $L(2) = Z(2) = 3$. Moreover, if $M > 2$ and if $Z(M)$ is odd, then M does not enter $\{L_n\}$.

Proof: $2|L_0$, so Theorem 1 implies $L(2) = Z(2) = 3$. If $M > 2$ and M enters $\{L_n\}$, then $M \nmid L_0$. Since $\{L_n\}$ is self-conjugate, Theorem 2 implies $2L(M) = Z(M)$, so $L(M) = \frac{1}{2}Z(M)$. Hence, when $M > 2$, M enters $\{L_n\}$ only when $Z(M)$ is even.

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Reference

1. Charles H. King. "Conjugate Generalized Fibonacci Sequences." *Fibonacci Quarterly* *6.1* (1968):46-49.

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Announcement

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