SETS OF TERMS THAT DETERMINE ALL THE TERMS OF A LINEAR RECURRENCE SEQUENCE

Clark Kimberling
University of Evansville, Evansville, IN 47222
(Submitted August 1989)

A second-order linear homogeneous recurrence sequence $u_0, u_1, u_2, \ldots$ is defined by a recurrence relation $u_n = au_{n-1} + bu_{n-2}$, where a and b are complex numbers with $b \neq 0$, and two initial terms u_0 and u_1. We raise the following question: for given a and b, what sets of terms, other than u_0 and u_1, are sufficient to determine the entire sequence? We shall see that *any* two terms are often sufficient, but not always. A comparable result will then be presented for recurrences of higher order.

Suppose a and b are given and v_p and v_q, where $p < q$, are known terms of a sequence satisfying $v_m = av_{m-1} + bv_{m-2}$. Then the terms u_0 and u_n of the sequence defined by $u_m = v_{m+p}$, where $n = q - p$, are known. Accordingly, without loss of generality, we recast the original question as follows: *under what conditions on a, b, and n do the values of u_0 and u_n determine the values of u_m for all $m \geq 0$?*

The answer depends on a sequence of bivariate polynomials defined recursively by $F_m(x, y) = xF_{m-1}(x, y) + yF_{m-2}(x, y)$, beginning with $F_1(x, y) = 1$ and $F_2(x, y) = x$. These are often called *Fibonacci polynomials*; indeed, $F_m(1, 1)$ is the m^{th} Fibonacci number.

Theorem 1: Suppose a and b are complex numbers satisfying $F_n(a, b) \neq 0$, where $b \neq 0$ and $F_n(x, y)$ denotes the Fibonacci polynomial of degree $n - 1$ in x. Then u_0 and u_n determine u_m for all $m \geq 0$.

Proof: If $n = 1$, then the recurrence $u_m = au_{m-1} + bu_{m-2}$ determines u_m inductively for all $m \geq 0$.

If $n = 2$, then the equation $u_2 = au_1 + bu_0$ yields $u_1 = (u_2 - bu_0)/a$, so that u_1 and hence all u_m are determined. [Note that $a \neq 0$, since $a = F_2(a, b)$.]

For $n \geq 3$, we have a system $u_s = au_{s-1} + bu_{s-2}$ of $n - 1$ equations, for $s = 2, 3, \ldots, n$. Write the first of these as $au_1 - u_2 = -bu_0$, the last as $bu_{n-2} + au_{n-1} = u_n$, and all the others as $bu_{s-2} + au_{s-1} - u_s = 0$. As an example, for $n = 5$, we have

$$\begin{aligned} au_1 - u_2 &= -bu_0 \\ bu_1 + au_2 - u_3 &= 0 \\ bu_2 + au_3 - u_4 &= 0 \\ bu_3 + au_4 &= u_5. \end{aligned}$$

The coefficient matrix of this system,

$$\begin{bmatrix} a & -1 & 0 & 0 \\ b & a & -1 & 0 \\ 0 & b & a & -1 \\ 0 & 0 & b & a \end{bmatrix}$$

clearly has determinant $F_5(a, b)$ given by Laplace expansion about the first column: $aF_4(a, b) + bF_3(a, b)$.

For the general case, $n \geq 3$, it easy to see, inductively, that the determinant of the coefficient matrix is $F_n(a, b)$. Accordingly, if $F_n(a, b) \neq 0$, then the system has a unique solution. In particular, u_1 is determined, so that u_m is determined for all $m \geq 0$.

Theorem 2: Suppose u_0 and u_n are known for some $n \geq 1$. Suppose, further, that a^2/b is a nonzero integer and one of the following holds:

(i) a^2/b does not equal -1, -2, or -3;
(ii) if $n \equiv 0 \bmod 3$, then $a^2 + b \neq 0$;
(iii) if $n \equiv 0 \bmod 4$, then $a^2 + 2b \neq 0$;
(iv) if $n \equiv 0 \bmod 6$, then $a^2 + 3b \neq 0$.

Then u_m is determined for all $m \geq 0$.

Proof: The polynomial $F_n(x, y)$ is an even function in x if n is odd, and odd in x if n is even. Accordingly, by the Fundamental Theorem of Algebra, this polynomial factors in the form

$$f_n(x^2, y) = (x^2 - c_1 y)(x^2 - c_2 y) \cdots (x^2 - c_{\left[\frac{n-1}{2}\right]} y)$$

if n is odd, and $xf_{n-1}(x^2, y)$ if n is even, where c_i is a complex number for $i = 1, 2, \ldots, n - 1$.

If a^2/b is a nonzero integer k, then $a^2 - kb = 0$, so that $c_i = a^2/b$ for some i. Thus, $x^2 - (a^2/b)y$ divides $F_n(x, y)$.

It is known ([1], Theorem 6) that the only divisors of $F_n(x, y)$ over the ring $I[x, y]$ that have degree 2 in x are the three second-degree Fibonacci-cyclotomic polynomials: $x^2 + y$, $x^2 + 2y$, $x^2 + 3y$, and that these are divisors if and only if n is divisible by 3, 4, or 6, respectively. Therefore, except for the four recognized cases, we have $F_n(a, b) \neq 0$, so that, by Theorem 1, u_m is determined for all $m \geq 0$.

Theorem 3: Suppose $a^2 + b = 0$ and u_0 is known. Then $u_m = (-a)^m u_0$ for every $m \equiv 0 \bmod 3$. Also, if u_k is known for some k not congruent to 0 modulo 3, then u_m is determined for all $m \geq 0$. In fact,

(1) $$u_m = (-a^3)^i u_j,$$

for $m = 3i + j$, $j = 0, 1, 2$, where $u_2 = au_1 - a^2 u_0$.

Proof: First, we shall establish equation (1). The statements

$$u_{3i} = (-1)^i a^{3i} u_0, \quad u_{3i+1} = (-1)^i a^{3i} u_1, \quad \text{and} \quad u_{3i+2} = (-1)^i a^{3i} u_2$$

are clearly true for $i = 0$. Assume them true for arbitrary $i \geq 0$. Then

$$\begin{aligned} u_{3i+3} &= au_{3i+2} + bu_{3i+1} \\ &= a(-1)^i a^{3i} u_2 - a^2(-1)^i a^{3i} u_1 \\ &= (-1)^i a^{3i+1}(u_2 - au_1) \\ &= (-1)^i a^{3i+1}(-a^2 u_0) \\ &= (-1)^{i+1} a^{3i+3} u_0, \end{aligned}$$

and, similarly,

$$u_{3i+4} = (-1)^{i+1} a^{3i+3} u_1 \quad \text{and} \quad u_{3i+5} = (-1)^{i+1} a^{3i+3} u_2.$$

By induction, therefore,

$$u_m = (-a^3)^i u_j \text{ for } m = 3i + j, \ j = 0, 1, 2.$$

Now equation (1) shows that u_0 determines those u_m for which m is a multiple of 3, and no others. However, if u_{3i+1} is also known for some i, then

$$u_{3i+1} = (-a^3)^i u_1,$$

so that u_1 is determined, and hence u_m is determined for all $m \geq 0$. A similar argument obviously applies if u_{3i+2} is known for some i.

Theorem 4: Suppose $a^2 + 2b = 0$ and u_0 is known. Then

$$u_m = (-1/4)^{m/4} a^m u_0 \text{ for every } m \equiv 0 \bmod 4.$$

If u_k is also known for some k not congruent to 0 modulo 4, then u_m is determined for all $m \geq 0$. In fact,

$$u_m = (-a^4/4)^i u_j \text{ for } m = 4i + j,\ j = 0,\ 1,\ 2,\ 3,$$

where $u_2 = au_1 - (a^2/2)u_0$ and $u_3 = (a^2/2)u_1 - (a^3/2)u_0$.

Proof: (The proof is similar to that of Theorem 3 and is omitted here.)

Theorem 5: Suppose $a^2 + 3b = 0$ and u_0 is known. Then

$$u_m = (-1/27)^{m/6} a^m u_0 \text{ for every } m \equiv 0 \bmod 6.$$

If u_k is also known for some k not congruent to 0 modulo 6, then u_m is determined for all $m \geq 0$. Explicitly,

$$u_2 = au_1 - (a^2/3)u_0,$$
$$u_3 = (2a^2/3)u_1 - (a^3/3)u_0,$$
$$u_4 = (a^3/3)u_1 - (2a^4/9)u_0,$$
$$u_5 = (a^4/9)u_1 - (a^5/9)u_0,$$

and $$u_m = (-a^6/27)^i u_j,$$

for $m = 6i + j = 0,\ 1,\ 2,\ 3,\ 4,\ 5$.

Proof: (The proof is similar to that of Theorem 3 and is omitted here.)

Second-order sequences for which $u_1 \neq 0$ and $u_0 = u_n = 0$ for some $n \geq 2$ are of special interest, since in this case $F_n(a, b) = 0$, so that Theorem 1 does not apply. Theorem 6 describes such sequences. [To see that $F_n(a, b) = 0$, note that the recurrence $u_m = au_{m-1} + bu_{m-2}$ gives

$$u_2 = au_1, \quad u_3 = au_2 + bu_1 = (a^2 + b)u_1 = u_1F_3(a, b),$$

and by induction, $u_n = u_1F_n(a, b)$.]

Theorem 6: Let $F_n(x, y)$ denote the n^{th} Fibonacci polynomial, where $n \geq 2$. If $u_1 = 0$ and $u_0 = u_n = 0$, then $F_n(a, b) = 0$, and there exist nonzero real numbers c, r and positive integers p, q such that

$$u_m = cr^m \sin mp\pi/q,$$

where n is an integer multiple of q, for $m = 0, 1, \ldots$.

Proof: From the Binet representation for the general term of a second-order homogeneous recurrence sequence,

$$u_m = w\alpha^m + z\overline{\alpha}^m.$$

It is easy to check that z must be a complex conjugate of w, so, after writing $w = a + bi$ and $\alpha = r(\cos\theta + i\sin\theta)$, we have

$$\begin{aligned} u_m &= (a + bi)r^m(\cos m\theta + i\sin m\theta) + (a - bi)r^m(\cos m\theta - i\sin m\theta) \\ &= 2r^m(a\cos m\theta - b\sin m\theta). \end{aligned}$$

Now a must equal 0, since $u_0 = 0$, and $\sin n\theta$ must equal 0, since $u_n = 0$. It follows that θ must be of the form $p\pi/q$, where n is a multiple of q. Thus, the asserted form for u_m has been demonstrated. Since u_m is not uniquely determined, Theorem 1 shows that $F\ (a, b) = 0$ (as was already proved differently just before the statement of Theorem 6).

Sequences of Higher Order

The method of proof of Theorem 1 extends readily to recurrence sequences of arbitrary order $k \geq 2$, as indicated by Theorem 7.

Theorem 7: Suppose $k \geq 2$, and suppose $c_0, c_1, \ldots, c_{k-1}$ are complex numbers satisfying $c_{k-1} \neq 0$. A set of k terms,

$$u_0,\ u_{m_1},\ u_{m_2},\ \ldots,\ u_{m_{k-1}},$$

where $0 < m_1 < m_2 < \cdots < m_{k-1}$, uniquely determine all the terms of a recurrence sequence given by

$$(2) \qquad u_n = c_{k-1}u_{n-1} + c_{k-2}u_{n-2} + \cdots + c_0 u_{n-k}$$

if and only if the matrix M defined below is nonsingular: let N denote the $(m_{k-1} - k + 1) \times (m_{k-1} + 1)$ matrix (a_{ij}) given by

$$a_{ij} = \begin{cases} c_{j-i+1} & \text{for } j = i-1, i, \ldots, i+k-2 \\ -1 & \text{for } j = i+k-1 \\ 0 & \text{for all remaining } j,\ 0 \leq j \leq m_{k-1}, \end{cases}$$

$$\text{for } i = 1, 2, \ldots, m_{k-1} - k + 1,$$

and define M to be the $(m_{k-1} - k + 1) \times (m_{k-1} - k + 1)$ matrix obtained by deleting from N the columns numbered $0, m_1, m_2, \ldots, m_{k-1}$.

Proof: Equation (2) generates, for $n = k, k+1, \ldots, m_{k-1}$, a system of $m_{k-1} - k + 1$ equations of the form

$$(3) \qquad c_{k-1}u_{n-1} + c_{k-2}u_{n-2} + \cdots + c_0 u_{n-k} - u_n = 0.$$

If all the terms $u_0, u_1, u_2, \ldots, u_{m_{k-1}}$ are regarded as unknowns, then the coefficient matrix of the system is N. If $u_0, u_{m_1}, u_{m_2}, \ldots, u_{m_{k-1}}$ are now regarded as known, and accordingly transposed to the right-hand side of each of the equations (3), then the coefficient matrix of the resulting system is M. By Cramer's Rule, this system has a unique solution if and only if $|M| \neq 0$.

As an example, consider a third-order recurrence

$$u_n = au_{n-1} + bu_{n-2} + cu_{n-3},$$

and suppose u_0, u_1, and u_m are known. (In the notation of Theorem 6, $k = 3$, $m_1 = 1$, and $m_2 = m$.) Define $T_1 = 1$, $T_2 = a$, and find for $m = 4$ that

$$N_4 = \begin{bmatrix} c & b & a & -1 & 0 \\ 0 & c & b & a & -1 \end{bmatrix},$$

which on deletion of columns 0, 1, and 4 leaves

$$M_4 = \begin{bmatrix} a & -1 \\ b & a \end{bmatrix}$$

with determinant $a^2 + b$. Define $T_3 = a^2 + b$. For $m = 5$,

$$N_5 = \begin{bmatrix} c & b & a & -1 & 0 & 0 \\ 0 & c & b & a & -1 & 0 \\ 0 & 0 & c & b & a & -1 \end{bmatrix} \text{ yields } M_5 = \begin{bmatrix} a & -1 & 0 \\ b & a & -1 \\ c & b & a \end{bmatrix},$$

with determinant $T_4 \equiv aT_3 + bT_2 + cT_1$. Continuing with $m = 6, 7, 8, \ldots$, we obtain recursively a sequence of trivariate polynomials:

$$T_m = aT_{m-1} + bT_{m-2} + cT_{m-1}.$$

Since, for example, $T_4(1, -1, 1) = 0$, Theorem 6 tells us that u_0, u_1, and u_5 are not sufficient to determine all the terms of a sequence obeying the recurrence $u_n = u_{n-1} - u_{n-2} + u_{n-3}$. On the other hand, as $T_5(1, -1, 1) \neq 0$, the terms u_0, u_1, and u_6 do determine the entire sequence.

Reference

1. C. Kimberling. "Generalized Cyclotomic Polynomials, Fibonacci Cyclotomic Polynomials, and Lucas Cyclotomic Polynomials." *Fibonacci Quarterly 18.2* (1980):108-26.

*****