A COMBINATORIAL INTERPRETATION OF THE SQUARE OF A LUCAS NUMBER

John Konvalina and Yi-Hsin Liu
University of Nebraska at Omaha, NE 68182
(Submitted October 1989)

1. Introduction

The Fibonacci numbers have a well-known combinatorial interpretation in terms of the total number of subsets of $\{1, 2, 3, \ldots, n\}$ not containing a pair of consecutive integers. Recently, Konvalina & Liu [4] showed that the squares of the Fibonacci numbers have a combinatorial interpretation in terms of the total number of subsets of $\{1, 2, 3, \ldots, 2n\}$ without unit separation. Two integers are called *uniseparate* if they contain exactly one integer between them. For example, the following pairs of integers are uniseparate: (1, 3), (2, 4), (3, 5), (4, 6), etc.

In this paper, we will show that the squares of the Lucas numbers also have a combinatorial interpretation in terms of subsets of $\{1, 2, \ldots, 2n\}$ without unit separation if the integers $\{1, 2, 3, \ldots, 2n\}$ are arranged in a circle instead of a line.

Let F_n denote the n^{th} Fibonacci number determined by the recurrence relation:

$$F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n \quad (n \geq 1).$$

Kaplansky [2] showed that the numbers of k-subsets of $\{1, 2, 3, \ldots, n\}$ not containing a pair of consecutive integers is

$$\binom{n+1-k}{k}.$$

Summing over all k-subsets, we obtain the well-known identity

$$(1) \qquad \sum_{k \geq 0} \binom{n+1-k}{k} = F_{n+2}.$$

Let $f(n, k)$ denote the number of k-subsets of $\{1, 2, 3, \ldots, n\}$ without unit separation. Konvalina [3] proved

$$(2) \qquad f(n, k) = \begin{cases} \displaystyle\sum_{i=0}^{[k/2]} \binom{n+1-k-2i}{k-2i} & \text{if } n \geq 2(k-1), \\ 0 & \text{if } n < 2(k-1). \end{cases}$$

Summing over all k-subsets, Konvalina & Liu [4] showed

$$(3) \qquad \sum_{k \geq 0} f(n, k) = \begin{cases} F_{m+2}^2 & \text{if } n = 2m, \\ F_{m+2}F_{m+3} & \text{if } n = 2m+1 \end{cases}$$

Next, let L_n denote the n^{th} Lucas number determined by the recurrence relation:

$$L_1 = 1,\ L_2 = 3,\ L_{n+2} = L_{n+1} + L_n \quad (n \geq 1).$$

The following identity expressing a Lucas number in terms of the sum of two Fibonacci numbers is well known (see Hoggatt [1]):

$$(4) \qquad L_n = F_{n+1} + F_{n-1}.$$

The Lucas numbers have a combinatorial interpretation in terms of the total number of subsets of $\{1, 2, 3, \ldots, n\}$ arranged in a circle and not containing a pair of consecutive integers (n and 1 are consecutive). One way to prove this is as follows: Kaplansky [2] showed that the number of k-subsets of $\{1, 2, 3, \ldots, n\}$ arranged in a circle and not containing a pair of consecutive integers is

$$(5) \qquad \frac{n}{k}\binom{n-k-1}{k-1} = \binom{n-k}{k} + \binom{n-k-1}{k-1}.$$

Summing over all k-subsets and applying (1) and (4), we obtain:

$$\sum_{k \ge 0} \frac{n}{k}\binom{n-k-1}{k-1} = \sum_{k \ge 0}\binom{n-k}{k} + \sum_{k \ge 0}\binom{n-k-1}{k-1} = F_{n+1} + F_{n-1} = L_n.$$

2. The Main Result

Let $g(n, k)$ denote the number of k-subsets of $\{1, 2, 3, \ldots, n\}$ arranged in a circle and without unit separation. Konvalina [3] proved the following identity:

$$(6) \qquad g(n, k) = f(n-2, k) + 2f(n-5, k-1) + 3f(n-6, k-2).$$

Let C_n denote the total number of subsets of $\{1, 2, 3, \ldots, n\}$ arranged in a circle and without unit separation. The following result relates the square of a Lucas number and C_{2m}.

Theorem: If $n > 2$, then

$$C_n = \begin{cases} L_m^2 & \text{if } n = 2m, \\ L_n & \text{if } n = 2m+1. \end{cases}$$

Proof: Summing over all k-subsets and applying (6), we have

$$(7) \qquad C_n = \sum_{k \ge 0} g(n, k) = \sum_{k \ge 0} f(n-2, k) + 2f(n-5, k-1) + 3f(n-6, k-2).$$

Even Case: $n = 2m$

Applying identity (3) to (7), we obtain

$$\begin{aligned} C_n &= \sum_{k \ge 0} f(n-2, k) + 2\sum_{k \ge 0} f(n-5, k-1) + 3\sum_{k \ge 0} f(n-6, k-2) \\ &= F_{m+1}^2 + 2F_{m-1}F_m + 3F_{m-1}^2 \\ &= F_{m+1}^2 + 2F_{m-1}(F_m + F_{m-1}) + F_{m-1}^2 \\ &= (F_{m+1} + F_{m-1})^2 = L_m^2. \end{aligned}$$

Odd Case: $n = 2m + 1$

Applying identity (3) to (7), we have

$$\begin{aligned} C_n &= F_{m+1}F_{m+2} + 2F_m^2 + 3F_{m-1}F_m \\ &= F_{m+1}F_{m+2} + 2F_m(F_m + F_{m-1}) + F_{m-1}F_m \\ &= F_{m+1}F_{m+2} + 2F_mF_{m+1} + F_{m-1}F_m \\ &= (F_{m+1}F_{m+2} + F_mF_{m+1}) + (F_mF_{m+1} + F_{m-1}F_m) \\ &= F_{2m+2} + F_{2m} = L_{2m+1} = L_n. \end{aligned}$$

Note: We have applied the following known identity (see [1], p. 59, identity I_{26} with $m = n - 1$): $F_{2n} = F_nF_{n+1} + F_{n-1}F_n$.

References

1. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969.
2. I. Kaplansky. Solution of the "Probleme des menages." *Bull. Amer. Math. Soc. 49* (1943):784-85.
3. J. Konvalina. "On the Number of Combinations without Unit Separation." *J. Combin. Theory, Ser. A 31* (1981):101-07.
4. J. Konvalina & Y.-H. Liu. "Subsets without Unit Separation and Products of Fibonacci Numbers." *Fibonacci Quarterly* (to appear).

*****