MULTIPLICATIVE PARTITIONS OF BIPARTITE NUMBERS

Bruce M. Landman
University of North Carolina at Greensboro, NC 27412

Raymond N. Greenwell
Hofstra University, Hempstead, NY 11550
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1. Introduction

For a positive integer n, let $f(n)$ be the number of multiplicative partitions of n. That is, $f(n)$ represents the number of different factorizations of n, where two factorizations are considered the same if they differ only in the order of the factors. For example, $f(12) = 4$, since $12 = 6 \cdot 2 = 4 \cdot 3 = 3 \cdot 2 \cdot 2$ are the four multiplicative partitions of 12. Hughes & Shallit [2] showed that $f(n) \le 2n^{\sqrt{2}}$ for all n. Mattics & Dodd [3] improved this to $f(n) \le n$, and in [4] they further improved this to $f(n) \le n/\log(n)$ for $n \ne 144$. In this paper, we generalize the notion of multiplicative partitions to bipartite numbers and obtain a corresponding bound.

By a j-partite number, we mean an ordered j-tuple $(n_1, \ldots, n_j)$, where all n_i are positive integers. Bipartite refers to the case $j = 2$. We can extend the idea of multiplicative partitions to bipartite numbers as follows. For positive integers m and n, define $f_2(m, n)$ to be the number of different ways to write (m, n) as a product $(a_1, b_1)(a_2, b_2)\ldots(a_k, b_k)$, where the multiplication is done coordinate-wise, all a_i and b_i are positive integers, $(1, 1)$ is not used as a factor of $(m, n) \ne (1, 1)$, and two such factorizations are considered the same if they differ only in the order of the factors. Hence, $(2, 1)(2, 1)(1, 4)$ and $(1, 4)(2, 1)(2, 1)$ are considered the same factorizations of $(4, 4)$, while $(2, 1)(2, 1)(1, 4)$ and $(1, 2)(1, 2)(4, 1)$ are considered different. Thus, for example, $f_2(6, 2) = 5$, since the five multiplicative partitions $(6, 2)$ are

$$(6, 2) = (6, 1)(1, 2) = (3, 2)(2, 1) = (3, 1)(2, 2)$$
$$= (3, 1)(2, 1)(1, 2).$$

It is clear that $f(n) = f_2(n, 1)$ for all n. In Section 2, we give an upper bound for $f_2(m, n)$. The definition of $f_2(m, n)$ may be extended to $f_j(n_1, \ldots, n_j)$ in an obvious way.

Throughout this paper, unless otherwise stated, $p_1 = 2$, $p_2 = 3$, ... will represent the sequence of primes.

2. An Upper Bound for $f_2(m, n)$

When first considering the function $f_2(m, n)$, some conjectures immediately came to mind:

(1) $f_2(m, n) = f(m)f(n)$ (2) $f_2(m, n) \le f(m)f(n)$
(3) $f_2(m, n) = f(mn)$ (4) $f_2(m, n) \le f(mn)$
(5) $f_2(m, n) \le mn/\log(mn)$ (6) $f_2(m, n) \le mn$.

Surprisingly, none of these is true. The values $f(2) = 1$, $f(6) = 2$, $f(12) = 4$, and $f_2(6, 2) = 5$ provide counterexamples to (1)-(5). As it turns out, (6) is also false (see Section 3).

In the next theorem, we establish an upper bound on $f_2(m, n)$. We will first need the following three lemmas.

Lemma 1: Let $\{p_1, \ldots, p_j\}$, $\{q_1, \ldots, q_k\}$, and $\{r_1, \ldots, r_{j+k}\}$ each be a set of distinct primes, and let

$$x = p_1^{a_1} \ldots p_j^{a_j}, \quad y = q_1^{b_1} \ldots q_k^{b_k}, \quad z = r_1^{a_1} \ldots r_j^{a_j} r_{j+1}^{b_1} \ldots r_{j+k}^{b_k},$$

where all a_i and b_i are positive integers. Then, $f(z) = f_2(x, y)$.

Proof: With each factorization

$$z = [r_1^{c_{11}} r_2^{c_{12}} \ldots r_{j+k}^{c_{1,j+k}}][r_1^{c_{21}} r_2^{c_{22}} \ldots r_{j+k}^{c_{2,j+k}}] \ldots [r_1^{c_{t1}} \ldots r_{j+k}^{c_{t,j+k}}]$$

we associate the following factorization of (x, y):

$$[p_1^{c_{11}} \ldots p_j^{c_{1j}}, q_1^{c_{1,j+1}} \ldots q_k^{c_{1,j+k}}] \ldots [p_1^{c_{t1}} \ldots p_j^{c_{tj}}, q_1^{c_{t,j+1}} \ldots q_k^{c_{t,j+k}}].$$

This association is obviously a one-to-one correspondence.

Lemma 1 can easily be extended to j-partite numbers. Thus, for example, $f_2(12, 4) = f(180) = f_3(4, 4, 2) = f_2(36, 2)$.

It is well known that

(a) $p_n > n \log n$ for $n \geq 1$, and
(b) $p_n < n(\log n + \log\log n)$ for $n \geq 6$ (see [5]).

As a consequence, we have the following lemma.

Lemma 2: For $n \geq 4$, $p_{2n-1}p_{2n} < p^{2.97}$.

Proof: Direct computation shows the inequality holds for $n = 4$, 5, and 6. Note that, for $n \geq 7$,

$$(2n - 1)(\log(2n - 1) + \log\log(2n - 1))2n(\log 2n + \log\log 2n) < (n \log n)^{2.97}.$$

Thus, from (a) and (b) above, $p_{2n-1}p_{2n} < (n \log n)^{2.97} < p^{2.97}$.

Lemma 3: Let $c_1 \geq c_2 \geq \cdots \geq c_k > 0$. Then

$$\prod_{i=1}^{k} [p_{2i-1}p_{2i}]^{c_i} < \prod_{i=1}^{k} p_i^{3.032c_i}.$$

Proof: If $k = 1$, the inequality holds, since $p_1p_2 < p_1^{2.585}$. For $k = 2$, since $p_3p_4 < p_2^{3.237}$, we have

$$[p_1p_2]^{c_1}[p_3p_4]^{c_2} < p_1^{2.585c_1}p_2^{3.237c_2}[p_1^{.4c_1}/p_2^{.252c_2}] = [p_1^{c_1}p_2^{c_2}]^{2.985}.$$

If $k = 3$,

$$(1) \qquad \prod_{i=1}^{3} [p_{2i-1}p_{2i}]^{c_i} < [p_1^{c_1}p_2^{c_2}]^{2.985}p_3^{3.084c_3}[p_1^{c_1}p_2^{c_2}]^{.047}/p_3^{.052c_3} = \prod_{i=1}^{3} p_i^{3.032c_i}.$$

If $k \geq 4$, the inequality follows easily from (1) and Lemma 2.

Theorem 1: Let m and n be positive integers with $(m, n) \neq (1, 1)$. Then

$$f_2(m, n) < (mn)^{1.516}/\log(mn).$$

Proof: We can assume that $m = p_1^{a_1} \ldots p_k^{a_k}$ and $n = p_1^{b_1} \ldots p_r^{b_r}$, where $k \geq r$ and $a_i \geq a_{i+1}$, $b_i \geq b_{i+1}$ for each i. Then, by Lemma 1,

$$f_2(m, n) = f(p_1^{a_1}p_2^{b_1}p_3^{a_2}p_4^{b_2} \ldots p_{2k-1}^{a_k}p_{2k}^{b_k}),$$

where $b_i = 0$ if $i > r$. For $i = 1, \ldots, k$, let

$$\alpha_i = \max\{a_i, b_i\}, \; \beta_i = \min\{a_i, b_i\}, \; c_i = (a_i + b_i)/2.$$

We first consider the case in which

$$\prod_{i=1}^{k} p_{2i-1}^{\alpha_i} p_{2i}^{\beta_i} \neq 144.$$

Then, by Lemmas 1 and 3 and the known bound for $f(n)$,

$$f_2(m, n) = f[p_1^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_4^{\beta_2} \ldots p_{2k-1}^{\alpha_k} p_{2k}^{\beta_k}] \le \frac{\prod_{i=1}^{k} [p_{2i-1}^{\alpha_i} p_{2i}^{\beta_i}]}{\log\left[\prod_{i=1}^{k} [p_{2i-1}^{\alpha_i} p_{2i}^{\beta_i}]\right]}$$

$$\le \frac{\prod_{i=1}^{k} [p_{2i-1}^{\alpha_i} p_{2i}^{\beta_i}]}{\log(mn)} \le \frac{\prod_{i=1}^{k} (p_{2i-1} p_{2i})^{c_i}}{\log(mn)} \le \frac{\prod_{i=1}^{k} p_i^{3.032 c_i}}{\log(mn)}$$

$$= \frac{\prod_{i=1}^{k} (p_i^{a_i + b_i})^{1.516}}{\log(mn)} = \frac{(mn)^{1.516}}{\log(mn)}.$$

In case $\prod_{i=1}^{k} p_{2i-1}^{\alpha_i} p_{2i}^{\beta_i} = 144$, it then follows by Lemma 1 that $mn \ge 2^6$. Noting that $f(144) = 29$, we see that the theorem is true in this case as well.

3. Remarks and Computations

3.1. Using the algorithm from [1], the values of $f_2(m, n)$ were found for all m and n such that $mn \le 2{,}000{,}000$ and for other selected values of m and n with mn as large as 167,961,600 by calculating the corresponding values of f as described in Lemma 1. Since large values of m and n tended to give the greatest values for the ratio $f_2(m, n)/mn$, and since these are the values that require the greatest computing time, we used the observations made in Remark 3.2 below to determine which pairs (m, n) to study.

3.2. Using the notation in the proof of Theorem 1, the pairs (m, n) can be described by the ordered $2k$-tuple $a_1b_1 \ldots a_kb_k$. In Table I below, we use this notation to list those $2k$-tuples we have found for which there exist ordered pairs (m, n) having ratios $r(a_1b_1 \ldots a_kb_k) = f_2(m, n)/mn > 1.5$ [given the $2k$-tuple, m and n are chosen so as to maximize $f_2(m, n)$].

TABLE I. Forms Yielding Large Ratios $f_2(m, n)/mn$

$a_1b_1 \ldots a_kb_k$	$f_2(m, n)$	$f_2(m, n)/(m, n)$
663311	162,075,802	2.17115
772211	61,926,494	1.86652
762211	30,449,294	1.83553
662211	15,173,348	1.82935
872211	119,957,268	1.80781
553311	33,439,034	1.79179
862211	58,256,195	1.75589
652211	7,126,811	1.71846
752211	14,096,512	1.69952
553211	10,511,373	1.68971
552211	3,400,292	1.63980
643311	30,428,542	1.63047
962211	107,097,889	1.61401
852211	26,610,876	1.60415
643211	9,584,844	1.54077
554411	255,339,989	1.52023
543311	14,162,812	1.51779

The prevalence of the forms $aabb11$ in the table is noteworthy. Although the forms $(a+1)(a-1)bb11$ also appear, the ratio is higher for $aabb11$. Similarly, the forms $(a+1)abb11$ have higher ratios than $(a+2)(a-1)bb11$. We suspect that sequences of the form $aabbcc11$ also have large ratios, but the lengthy computation time made this infeasible to verify. A result which helps explain the prevalence of trailing 1's in the sequence $a_1b_1 \ldots a_kb_k$ is as follows: Let

$$j = \begin{cases} 2k & \text{if } b_k \neq 0 \\ 2k-1 & \text{if } b_k = 0 \end{cases}$$

and let $c_1 \ldots c_j$ denote $a_1b_1 \ldots a_kb_k$. Then, if $1 \le i \le 2k$,

$$[6p_{[(i+2)/2]}/5p_{[(j+1)/2]}] \;\; r(c_1 \ldots c_i \ldots c_j)$$
$$\le \;\; r(c_1 \ldots c_{i-1}c_{i+1} \ldots c_j1) \text{ when } c_i \ge 2,$$

where [ ] denotes the greatest integer function. This result follows easily from the lemma on page 22 of [1].

3.3. For the more general function $f_j(n_1, \ldots, n_j)$, note that

$$f_j(q_1, \ldots, q_j) = f(p_1, \ldots, p_j) = B(j),$$

where $B(j)$ is the j^{th} Bell number and the q_i are any primes. ($B(j)$ grows very fast. See, e.g., [6].)

3.4. If we set

$$f_2(m, n) = (mn)^{\alpha}/\log(mn),$$

then, for all m and n for which $f_2(m, n)$ was calculated, $\alpha < 1.251$. The largest value of α occurred when $m = n = 24$ with $f_2(24, 24) = 444$. (This was the only case in which $\alpha > 5/4$.) Based on these data, we propose the following

Conjecture: $f_2(m, n) < (mn)^{1.251}/\log(mn)$ for all m and n.

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