SOME CONVOLUTION-TYPE AND COMBINATORIAL IDENTITIES PERTAINING TO BINARY LINEAR RECURRENCES

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(Submitted September 1989)

Introduction

Let sequences $\{r_n\}$ and $\{s_n\}$ be defined for $n \geq 0$. Letting

$$t_n = \sum_{k=0}^{n} r_k s_{n-k},$$

we obtain a sequence $\{t_n\}$ which is called the convolution of $\{r_n\}$ and $\{s_n\}$. In keeping with the ideas of V. E. Hoggatt, Jr. [7], one may define iterated convolution sequences as follows:

$$r_n^{(0)} = r_n; \quad r_n^{(j)} = \sum_{k=0}^{n} r_k r_{n-k}^{(j-1)} \quad \text{for } j \geq 1.$$

In particular, if $\{F_n\}$ denotes the Fibonacci sequence, then

$$F_n^{(1)} = \sum_{k=0}^{n} F_k F_{n-k}$$

is the convolution of the Fibonacci sequence with itself. Hoggatt [7] obtained the generating function:

$$x/(1 - x - x^2)^{j+1} = \sum_{n=0}^{\infty} F_n^{(j)} x^n.$$

The convolved sequence $F_n^{(1)}$ was also considered by Bicknell [2] and by Hoggatt & Bicknell-Johnson [8]. For related results, see also Bergum & Hoggatt [1] and Horadam and Mahon [9].

Let primary and secondary binary linear recurrences be defined, respectively, by

$$u_0 = 0,\ u_1 = 1,\ u_n = Pu_{n-1} - Qu_{n-2} \text{ for } n \geq 2;$$

$$v_0 = 2,\ v_1 = P,\ v_n = Pv_{n-1} - Qv_{n-2} \text{ for } n \geq 2,$$

where P and Q are nonzero, relatively prime integers such that $D = P^2 - 4Q \neq 0$. In this paper, we generalize prior results of Hoggatt and others by developing formulas for weighted convolutions of the type

$$\sum_{k=0}^{n} f(n, k) r_k s_{n-k},$$

where each of r_n and s_n is u_n or v_n and the weight function $f(n, k)$ is defined for $n \geq 0$ and $0 \leq k \leq n$ and satisfies the symmetry condition

$$f(n, n-k) = f(n, k) \text{ for all } k.$$

In addition, we prove some results about the sums

$$\sum_{k=0}^{n} \binom{n}{k} u_k \quad \text{and} \quad \sum_{k=0}^{n} \binom{n}{k} v_k.$$

Preliminaries

Let the roots of the equation: $t^2 - Pt + Q = 0$ be

$$a = \tfrac{1}{2}(P + D^{\frac{1}{2}}), \quad b = \tfrac{1}{2}(P - D^{\frac{1}{2}}),$$

so that

(1) $a + b = P$

(2) $ab = Q$

(3) $a - b = D^{\frac{1}{2}}$

(4) $u_n = (a^n - b^n)/(a - b)$

(5) $v_n = a^n + b^n$

(6) $v_n = 2u_{n+1} - Pu_n$

(7) $v_n = Pu_n - 2Qu_{n-1}$

(8) $v_n = u_{n+1} - Qu_{n-1}$

(9) $t/(1 - Pt + Qt^2) = \sum_{n=0}^{\infty} u_n t^n$

(10) $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

(11) $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

(12) $\sum_{k=0}^{n} k(n - k) = \dfrac{n^3 - n}{6}$

(13) $\sum_{k=0}^{n} k^2(n - k)^2 = \dfrac{n^5 - n}{30}$

The Main Results

Theorem 1:

(a) $$\sum_{k=0}^{n} u_k u_{n-k} = \frac{(n + 1)v_n - 2u_{n+1}}{D} = \frac{nv_n - Pu_n}{D} = \frac{(n - 1)v_n - 2Qu_{n-1}}{D}$$

(b) $$\sum_{k=0}^{n} \binom{n}{k} u_k u_{n-k} = \frac{2^n v_n - 2P^n}{D}$$

Proof: Without restriction on $f(n, k)$, (4) implies

$$\sum_{k=0}^{n} f(n, k) u_k u_{n-k} = \sum_{k=0}^{n} f(n, k)\left(\frac{a^k - b^k}{a - b}\right)\left(\frac{a^{n-k} - b^{n-k}}{a - b}\right)$$

$$= (a - b)^{-2} \sum_{k=0}^{n} f(n, k)(a^n + b^n - a^k b^{n-k} - a^{n-k} b^k)$$

$$= D^{-1}\left(v_n \sum_{k=0}^{n} f(n, k) - \sum_{k=0}^{n} f(n, k)(a^k b^{n-k} + a^{n-k} b^k)\right),$$

using (3) and (5).

(a) If $f(n, k) = 1$, we get

$$\sum_{k=0}^{n} u_k u_{n-k} = D^{-1}\left((n + 1)v_n - \sum_{k=0}^{n}(a^k b^{n-k} + a^{n-k}b^k)\right).$$

Now

$$\sum_{k=0}^{n} a^k b^{n-k} = \sum_{k=0}^{n} a^{n-k}b^k = b^n\left(\frac{(a/b)^{n+1} - 1}{(a/b) - 1}\right) = \frac{a^{n+1} - b^{n+1}}{a - b} = u_{n+1},$$

so

$$\sum_{k=0}^{n} u_k u_{n-k} = \frac{(n + 1)v_n - 2u_{n+1}}{D}.$$

The other parts of (a) follow from (6) and (7), since

$$(n + 1)v_n - 2u_{n+1} = nv_n + v_n - 2u_{n+1} = nv_n - Pu_n$$
$$= (n - 1)v_n + v_n - Pu_n = (n - 1)v_n - 2Qu_{n-1}.$$

(b) If $f(n, k) = \binom{n}{k}$, we get

$$\sum_{k=0}^{n}\binom{n}{k}u_k u_{n-k} = D^{-1}\left(v_n\sum_{k=0}^{n}\binom{n}{k} - \sum_{k=0}^{n}\binom{n}{k}(a^{n-k}b^k + a^k b^{n-k})\right)$$
$$= D^{-1}(2^n v_n - 2(a + b)^n) = \frac{2^n v_n - 2P^n}{D}$$

using (1) and (10).

Theorem 2:

(a) $$\sum_{k=0}^{n} v_k v_{n-k} = (n + 1)v_n + 2u_{n+1}$$

(b) $$\sum_{k=0}^{n}\binom{n}{k}v_k v_{n-k} = 2^n v_n + 2P^n.$$

Proof: The proof is similar to that of Theorem 1, except that we use (5) instead of (4).

Theorem 3:

$$w_{n-1} = \sum_{k=0}^{n} u_k u_{n-k} \text{ for } n \ge 1$$

if and only if $w_0 = 0$, $w_1 = 1$, $w_n = Pw_{n-1} - Qw_{n-2} + u_n$ for $n \ge 2$.

Proof: (Sufficiency) Following Carlitz [4], let

$$W(t) = \sum_{n=0}^{\infty} w_n t^n.$$

Then

$$(1 - Pt + Qt^2)W(t) = w_0 + (w_1 - Pw_0)t + \sum_{n=2}^{\infty}(w_n - Pw_{n-1} + Qw_{n-2})t^n$$
$$= t + \sum_{n=2}^{\infty} u_n t^n = \sum_{n=0}^{\infty} u_n t^n = t/(1 - Pt + Qt^2),$$

so $W(t) = t/(1 - Pt + Qt^2)^2$, from which it follows by (9) that

$$w_{n-1} = \sum_{k=0}^{n} u_k u_{n-k}.$$

(Necessity) (Induction on n). Let

$$w_{n-1} = \sum_{k=0}^{n} u_k u_{n-k} \text{ for } n \geq 1.$$

By direct evaluation, we have

$$w_0 = 0,\ w_1 = 1,\ w_2 = 2P,\ w_3 = 3P^2 - 2Q.$$

Theorem 1(a) implies $w_{n-1} = D^{-1}(nv_n - Pu_n)$. Now

$$Pw_1 - Qw_0 + u_2 = P(1) - Q(0) + P = 2P = w_2;$$

$$Pw_2 - Qw_1 + u_3 = P(2P) - Q(1) + (P^2 - Q) = 3P^2 - 2Q = w_3.$$

$$\begin{aligned} Pw_{n-1} - Qw_{n-2} &= \frac{P}{D}(nv_n - Pu_n) - \frac{Q}{D}((n-1)v_{n-1} - Pu_{n-1}) \\ &= \frac{1}{D}(Pv_n + (n-1)(Pv_n - Qv_{n-1}) - P(Pu_n - Qu_{n-1})) \\ &= \frac{1}{D}(Pv_n + (n-1)v_{n+1} - Pu_{n+1}) \\ &= \frac{1}{D}(Pv_n - 2v_{n+1} + (n+1)v_{n+1} - Pu_{n+1}) \\ &= w_n - \frac{1}{D}(2v_{n+1} - Pv_n). \end{aligned}$$

But $2v_{n+1} - Pv_n = 2(a^{n+1} + b^{n+1}) - (a+b)(a^n + b^n) = a^{n+1} + b^{n+1} - ab^n - a^n b = (a^n - b^n)(a - b) = Du_n$. Therefore,

$$Pw_{n-1} - Qw_{n-2} + u_n = w_n - \frac{1}{D}(Du_n) + u_n = w_n.$$

Theorem 4: If

$$x_n = \sum_{k=0}^{n} v_k v_{n-k} \text{ for } n \geq 0,$$

then

$$x_0 = 4,\ x_1 = 4P,\ x_n = Px_{n-1} - Qx_{n-2} + Du_n \text{ for } n \geq 2.$$

Proof: This is similar to the proof of Necessity in Theorem 3, and therefore is omitted here.

Lemma 1: Let $f(n, k)$ be a function such that $f(n, n-k) = f(n, k)$ for all k such that $0 \leq k \leq n$, where n and k are nonnegative integers. Then

$$\sum_{k=0}^{n} Q^k f(n, k) u_{n-2k} = 0.$$

Proof: Let

$$S_n = \sum_{k=0}^{n} Q^k f(n, k) u_{n-2k},\quad n^* = [\tfrac{1}{2}(n-1)],\quad S_1 = \sum_{k=0}^{n^*} Q^k f(n, k) u_{n-2k}.$$

Then

$$S_n - S_1 = \sum_{j=n-n^*}^{n^*} Q^j f(n, j) u_{n-2j}.$$

Letting $k = n - j$, we obtain

$$S_n - S_1 = \sum_{k=0}^{n^*} Q^{n-k} f(n, n-k) u_{2k-n} = \sum_{k=0}^{n^*} f(n, k) Q^{n-k}(-u_{n-2k}/Q^{n-2k}),$$

by (14), that is,

$$S_n - S_1 = -\sum_{k=0}^{n*} Q^k f(n, k) u_{n-2k} = -S_1, \text{ so } S_n = 0.$$

Theorem 5: If $f(n, k)$ satisfies the hypothesis of Lemma 1 above, then

$$\sum_{k=0}^{n} f(n, k) u_k v_{n-k} = u_n \left(\sum_{k=0}^{n} f(n, k) \right).$$

Proof:
$$\begin{aligned}\sum_{k=0}^{n} f(n, k) u_k v_{n-k} &= \sum_{k=0}^{n} f(n, k) \left(\frac{a^k - b^k}{a - b} \right) (a^{n-k} + b^{n-k}) \\ &= \sum_{k=0}^{n} f(n, k) \left(\frac{a^n - b^n - a^{n-k}b^k + a^k b^{n-k}}{a - b} \right) \\ &= \sum_{k=0}^{n} f(n, k) \left(u_n - (ab)^k \left(\frac{a^{n-2k} - b^{n-2k}}{a - b} \right) \right) \\ &= u_n \left(\sum_{k=0}^{n} f(n, k) \right) - \sum_{k=0}^{n} Q^k f(n, k) u_{n-2k} = u_n \left(\sum_{k=0}^{n} f(n, k) \right),\end{aligned}$$

by Lemma 1.

Corollary 1:

(a) $\displaystyle\sum_{k=0}^{n} u_k v_{n-k} = (n + 1) u_n$ (b) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} u_k v_{n-k} = 2^n u_n$

(c) $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 u_k v_{n-k} = \binom{2n}{n} u_n$ (d) $\displaystyle\sum_{k=0}^{n} k(n - k) u_k v_{n-k} = \left(\frac{n^3 - n}{6} \right) u_n$

(e) $\displaystyle\sum_{k=0}^{n} k^2 (n - k)^2 u_k v_{n-k} = \left(\frac{n^5 - n}{30} \right) u_n$

Proof: This follows from Theorem 5 and (10) through (13).

Theorem 6: Let u_n and v_n be the primary and secondary binary linear recurrences, respectively, with parameters P and Q, as defined in the introduction, and with discriminant $D = P^2 - 4Q$. Define

$$U_n = \sum_{k=0}^{n} \binom{n}{k} u_k, \quad V_n = \sum_{k=0}^{n} \binom{n}{k} v_k.$$

Then, U_n and V_n are also primary and secondary binary linear recurrences, respectively, with parameters $P* = P + 2$, $Q* = P + Q + 1$, and discriminant $D* = D$.

Proof:
$$\begin{aligned}U_n &= \sum_{k=0}^{n} \binom{n}{k} u_k = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{a^k - b^k}{a - b} \right) = D^{-\frac{1}{2}} \left(\sum_{k=0}^{n} \binom{n}{k} a^k - \sum_{k=0}^{n} \binom{n}{k} b^k \right) \\ &= \frac{(a + 1)^n - (b + 1)^n}{a - b} = \frac{(a + 1)^n - (b + 1)^n}{(a + 1) - (b + 1)}.\end{aligned}$$

If we let $A = a + 1$, $B = b + 1$, then

$$U_n = \frac{A^n - B^n}{A - B},$$

a primary binary linear recurrence with parameters

$$P* = A + B = (a + 1) + (b + 1) = (a + b) + 2 = P + 2,$$

and

$$Q^* = AB = (a + 1)(b + 1) = ab + (a + b) + 1 = P + Q + 1.$$

Similarly, if

$$V_n = \sum_{k=0}^{n} \binom{n}{k} v_k,$$

then $V_n = A^n + B^n$, a secondary binary linear recurrence with A and B as above. Furthermore,

$$D^* = (P^*)^2 - 4Q^* = (P + 2)^2 - 4(P + Q + 1)$$
$$= P^2 + 4P + 4 - 4P - 4Q - 4 = P^2 - 4Q = D.$$

Theorem 7: Let $\{u_n\}$ and $\{v_n\}$ be primary and secondary binary linear recurrences with discriminant $D > 0$. Then there exists a positive integer, m, such that

$$\sum_{k=0}^{n} \binom{n}{k} u_k = u_{mn}, \quad \sum_{k=0}^{n} \binom{n}{k} v_k = v_{mn}$$

if and only if $m = 2$, $u_n = F_n$, $v_n = L_n$.

Proof: To prove sufficiency, we note that, if $P = -Q = 1$, so that $u_n = F_n$, $v_n = L_n$, then $a^2 = a + 1 = A$, $b^2 = b + 1 = B$, so that Theorem 6 implies

$$\sum_{k=0}^{n} \binom{n}{k} u_k = \frac{A^n - B^n}{A - B} = \frac{a^{2n} - b^{2n}}{a - b} = u_{2n},$$

$$\sum_{k=0}^{n} \binom{n}{k} v_k = A^n + B^n = a^{2n} + b^{2n} = v_{2n}.$$

To prove necessity, using the notation of Theorem 6, we note that hypothesis, (4), and (5) imply

$$\frac{A^n - B^n}{a - b} = \frac{a^{mn} - b^{mn}}{a - b}, \quad A^n + B^n = a^{mn} + b^{mn}.$$

Therefore, $A = a^m$, $B = b^m$, so that $a^m = a + 1$, $b^m = b + 1$. Let

$$f_m(x) = x^m - x - 1.$$

Then $f_m(a) = f_m(b) = 0$. If m is odd, then $f_m(x)$ has critical values at

$$x = \pm m^{[-1/(m-1)]}.$$

It is easily verified that $f_m(\pm m^{[-1/(m-1)]}) < 0$. Therefore, $f_m(x)$ has a unique real root, so $a = b$, which implies $D = 0$, contrary to hypothesis. If m is even, then $f_m(x)$ has a minimum at $x = m^{[-1/(m-1)]}$, and $f_m(m^{[-1/(m-1)]}) < 0$, so $f_m(x)$ has two real roots a and b with $a > b$. Now,

$$f_m(-1) = 1, \ f_m(0) = f_m(1) = -1, \ f_m(2) = 2^m - 3 > 0, \text{ for } m \geq 2,$$

so we must have $-1 < b < 0$ and $1 < a < 2$. Therefore, $0 < a + b < 2$ and $-2 < ab < 0$. Since $a + b$ and ab must be integers, we have $P = a + b = 1$, $Q = ab = -1$. It now follows that $u_n = F_n$, $v_n = L_n$, $a^m = a + 1 = a^2$, so $m = 2$.

Concluding Remarks

If $P = -Q = 1$, then $D = 5$, $u_n = F_n$, and $v_n = L_n$ (the n^{th} Lucas number). In this case, Theorems 1(a), 1(b), 2(a), 2(b), say, respectively:

$$\text{(I)} \quad \sum_{k=0}^{n} F_k F_{n-k} = \frac{nL_n - F_n}{5}$$

(II) $\sum_{k=0}^{n}\binom{n}{k}F_k F_{n-k} = \frac{2^n L_n - 2}{5}$

(III) $\sum_{k=0}^{n} L_k L_{n-k} = (n+1)L_n + 2F_{n+1}$

(IV) $\sum_{k=0}^{n}\binom{n}{k}L_k L_{n-k} = 2^n L_n + 2$

(I) was obtained by Hoggatt & Bicknell-Johnson [8]; an alternate form of (I) was given by Knuth [10]; (I) and (II) appeared without proof in Wall [11]; (II) and (IV) were given by Buschman [3].

Theorem 7 also yields the identities

(V) $\sum_{k=0}^{n}\binom{n}{k}F_k = F_{2n}$; $\sum_{k=0}^{n}\binom{n}{k}L_k = L_{2n}$.

(V) appeared in papers by Gould [6] and by Carlitz & Ferns [5].

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