SUMMING INFINITE SERIES WITH SEX

Herb Silverman
College of Charleston, Charleston, SC 29424
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The usual way of computing the sum of the series $\sum_{n=1}^{\infty} nx^n$ for particular choices of x, $|x| < 1$, is to start with the geometric series

$$(1) \qquad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

and then appeal to uniform convergence and interval of convergence properties to obtain

$$x \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=1}^{\infty} nx^n = x \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2}.$$

What follows is a more insightful proof for $0 < x < 1$ that is accessible to students in finite mathematics classes who are familiar with neither infinite series nor calculus.

The expected value of a finite random variable $X = \{x_1, \ldots, x_n\}$ with associated probabilities $\{f(x_1), \ldots, f(x_n)\}$ is

$$\sum_{i=1}^{n} x_i f(x_i).$$

Consider the problem of determining the number of children a couple would expect to have if they continued to reproduce until a girl was born. The probability of having exactly n children would be

$$\left(\frac{1}{2}\right)^{n-1}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^n,$$

which means that

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

children would be expected. But, since *exactly one* girl is expected and a boy is as likely as a girl, this sum must be "two." More generally, suppose the probability of a boy is x and of a girl is $1 - x$. Then, for every girl, we would expect $x/(1 - x)$ boys, so the expected number of children $x/(1 - x) + 1$ could be expressed as

$$\sum_{n=1}^{\infty} nx^{n-1}(1-x) = \frac{x}{1-x} + 1 = \frac{1}{1-x},$$

from which we conclude that

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

One could similarly find different sums by specifying other gender restrictions. For instance, the probability of the k^{th} girl being the n^{th} child is

$$\binom{n-1}{k-1}x^{n-k}(1-x)^k.$$

Therefore,

$$\sum_{n=k}^{\infty} n\binom{n-1}{k-1}x^{n-k}(1-x)^k = k\left(\frac{x}{1-x}+1\right) = \frac{k}{1-x}$$

or, equivalently,

$$(2) \qquad \sum_{n=k}^{\infty}\binom{n}{k}x^n = \frac{x^k}{(1-x)^{k+1}}.$$

Since the probability of a girl being born

$$\left(\sum_{n=1}^{\infty}x^{n-1}(1-x)\right)$$

must be "one," we have an alternate proof of (1). Note that (2) may also be established by differentiating k times the identity (1).

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