ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-459 *Proposed by Stanley Rabinowitz, Westford, MA*

Prove that for all $n > 3$,

$$\frac{13\sqrt{5} - 19}{10}L_{2n+1} + 4.4(-1)^n$$

is very close to the square of an integer.

H-460 *Proposed by H.-J. Seiffert, Berlin, Germany*

Define the Fibonacci polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_{n+2}(x) = xF_{n+1}(x) + F_n(x).$$

Show that, for all positive reals x,

(a) $$\sum_{k=1}^{n-1} 1/\left(x^2 + \sin^2 \frac{k\pi}{2n}\right) = \frac{(2n-1)F_{2n+1}(2x) + (2n+1)F_{2n-1}(2x)}{4x(x^2+1)F_{2n}(2x)} - \frac{1}{2x^2},$$

(b) $$\sum_{k=1}^{n-1} 1/\left(x^2 + \sin^2 \frac{k\pi}{2n}\right) \sim n/(x\sqrt{x^2+1}),\ \text{as } n \to \infty,$$

(c) $$\sum_{k=1}^{n-1} 1/\sin^2 \frac{k\pi}{2n} = 2(n^2-1)/3.$$

H-461 *Proposed by Lawrence Somer, Washington, D.C.*

Let $\{u_n\} = u(a, b)$ denote the Lucas sequence of the first kind satisfying the recursion relation

$$u_{n+2} = au_{n+1} + bu_n,$$

where a and b are nonzero integers and the initial terms are $u_0 = 0$ and $u_1 = 1$. The prime p is a primitive divisor of u_n if $p \mid u_n$ but $p \nmid u_m$ for $1 \le m \le n-1$. It is known (see [1], p. 200) for the Fibonacci sequence $\{F_n\} = u(1, 1)$ that, if p is an odd prime divisor of F_{2n+1}, where $n \ge 1$, then $p \equiv 1 \pmod 4$.

(i) Find an infinite number of recurrences $u(a, b)$ such that every odd primitive prime divisor p of any term of the form u_{2n+1} or u_{4n} satisfies $p \equiv 1 \pmod 4$, where $n \geq 1$.

(ii) Find an infinite number of recurrences $u(a, b)$ such that every odd primitive prime divisor p of any term of the form u_{4n} or u_{4n+2} satisfies $p \equiv 1 \pmod 4$, where $n \geq 1$.

Reference

1. E. Lucas. "Théorie des Fonctions Numériques Simplement Périodiques." *Amer. J. Math.* *1* (1878):184-240, 289-321.

SOLUTION

Either Way

H-441 *Proposed by Albert A. Mullin, Huntsville, AL (Vol. 28, no. 2, May 1990)*

By analogy with palindrome, a *validrome* is a sentence, formula, relation, or verse that remains valid whether read forward or backward. For example, relative to *prime* factorization, 341 is a factorably validromic number since $341 = 11 \cdot 31$, and when backward gives $13 \cdot 11 = 143$, which is also correct. (1) What is the largest factorably validromic square you can find? (2) What is the largest factorably validromic square, *avoiding* palindromic numbers, you can find? Here are three examples of factorably validromic squares:

$$13 \cdot 13, \quad 101 \cdot 101, \quad 311 \cdot 311.$$

Solution by Paul S. Bruckman, Edmonds, WA

Suppose $n = \theta_1\theta_2\ldots\theta_r$ is in denary notation; we write

$$n' = \theta_r\theta_{r-1}\ldots\theta_1.$$

Given two natural numbers m and n, we say the product $m \times n$ is *validromic* if and only if $m \times n = m' \times n'$. A natural number n is said to be a *validromic square root* if and only if:

(1) $\quad (n^2)' = (n')^2.$

Let V denote the set of validromic square roots; we also write $n \in V$ if equation (1) holds. In this case, we also call n^2 a *validromic square*.

Some interesting properties of such numbers are derived by analyzing the familiar "long multiplication" process, somewhat modified. The multiplication for $n^2 = n \times n$ is indicated below:

$$(2)\qquad \begin{array}{cccccccccc}
 & & & & & \theta_1 & \theta_2 & \cdots & \theta_r & \\
 & & & & \times & \theta_1 & \theta_2 & \cdots & \theta_r & \\
\hline
 & & & & & \theta_1\theta_r & \theta_2\theta_r & \cdots & \theta_{r-1}\theta_r & \theta_r^2 \\
 & & & & \theta_1\theta_{r-1} & \theta_2\theta_{r-1} & \theta_3\theta_{r-1} & \cdots & \theta_r\theta_{r-1} & \\
 & & & \cdot^{\cdot^{\cdot}} & \vdots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} & & \\
 & & \theta_1\theta_2 & \cdots & \theta_{r-2}\theta_2 & \theta_{r-1}\theta_2 & \theta_r\theta_2 & & & \\
 & \theta_1^2 & \theta_2\theta_1 & \cdots & \theta_{r-1}\theta_1 & \theta_r\theta_1 & & & & \\
\hline
 & s_1 & s_2 & \cdots\ s_{r-1} & & s_r & s_{r+1} & \cdots & s_{2r-2} & s_{2r-1} \\
a_0 & a_1 & a_2 & \cdots\ a_{r-1} & & a_r & a_{r+1} & \cdots & a_{2r-2} & a_{2r-1}
\end{array}$$

In this product, the terms $\theta_i\theta_j$ are *not* reduced (mod 10) as they would normally be, nor are the columnar sums s_k. Therefore,

$$s_k = \sum_{\substack{i+j=k+1\\ 1\le i,\ j\le r}} \theta_i\theta_j, \text{ or more precisely,}$$

$$(3)\qquad s_k = \sum_{i=\max(k-r+1,\,1)}^{\min(k,\,r)} \theta_i\theta_{k+1-i},\ k = 1, 2, \ldots, 2r-1.$$

Thus, the terms $\theta_i\theta_j$ and the sums s_k are not necessarily denary digits. However, the a_k's (indicated below the s_k's) *are* denary digits, obtained by the process of "carrying forward and bringing down" familiar to any schoolchild. We do not preclude the possibility $a_0 = 0$.

Next, we carry out the similar multiplication for $(n')^2 = n'\times n'$:

$$(4)\qquad \begin{array}{cccccccccc}
 & & & & & \theta_r & \theta_{r-1} & \cdots & \theta_1 & \\
 & & & & & \times\ \theta_r & \theta_{r-1} & \cdots & \theta_1 & \\
\cline{6-10}
 & & & & & \theta_r\theta_1 & \theta_{r-1}\theta_1 & \cdots & \theta_2\theta_1 & \theta_1^2 \\
 & & & & \theta_r\theta_2 & \theta_{r-1}\theta_2 & \theta_{r-2}\theta_2 & \cdots & \theta_1\theta_2 & \\
 & & & ⋰ & \vdots & \vdots & \vdots & ⋰ & & \\
 & & \theta_r\theta_{r-1} & \cdots & \theta_3\theta_{r-1} & \theta_2\theta_{r-1} & \theta_1\theta_{r-1} & & & \\
 & \theta_r^2 & \theta_{r-1}\theta_r & \cdots & \theta_2\theta_r & \theta_1\theta_r & & & & \\
\hline
 & s_{2r-1} & s_{2r-2} & \cdots & s_{r+1} & s_r & s_{r-1} & \cdots & s_2 & s_1 \\
b_0 & b_1 & b_2 & \cdots & b_{r-1} & b_r & b_{r+1} & \cdots & b_{2r-2} & b_{2r-1}
\end{array}$$

As in the first product, we allow $b_0 = 0$. The observation that the columnar sums s_k in (4) are identical to those in (2) (except in reverse order) is a consequence of their consisting of the same components $\theta_i\theta_j$, albeit in permuted order. In fact, we see that if we reverse the order of the r "product-rows" in (4), then reverse the order of the digits in each such row, we obtain the corresponding product-rows of (2).

Using the notation introduced, we call the product $n\times n$ *proper* if and only if, for all $i, j\in\{1, 2, \ldots, r\}$, $k\in\{1, 2, \ldots, 2r-1\}$, the products $\theta_i\theta_j$ and the s_k's are all denary digits. Otherwise, we say that the product $n\times n$ is *improper*. We now prove a useful characterization of validromic square roots.

Theorem 1: $n\in V$ if and only if $n\times n$ is proper.

Proof: First, suppose $n\times n$ is proper. Looking at (2) and (4), it is clear that $a_0 = b_0 = 0$, and moreover that $a_k = s_k = b_{2r-k}$, $k = 1, 2, \ldots, 2r-1$. Equivalently, $(n^2)' = (n')^2$, or $n\in V$.

Conversely, suppose that $s_k \ne a_k$ for some k. Let $s_k = a_k$, for all $k > h$, but $s_h = a_h + 10d_h$ for some $d_h > 1$. Inspection of (4) yields:

$$b_{2r-h} = a_h, \text{ but } b_{2r-h-1} \equiv s_{h+1} + d_h \pmod{10};$$

if $h = 2r-1$, we define $s_{2r} = a_{2r} = 0$. If $d_h \not\equiv 0 \pmod{10}$, then

$$b_{2r-h-1} \equiv a_{h+1} + d_h \pmod{10}, \text{ so } b_{2r-h-1} \ne a_{h+1}.$$

If $d_h \equiv 0 \pmod{10}$, then

$$b_{2r-h-1} = a_{h+1}, \text{ but } b_{2r-h-2} \equiv s_{h+2} + d_{h+1} \equiv a_{h+2} + d_{h+1} \pmod{10},$$

where $d_{h+1} = d_h/10$. We apply the same argument until we find a nonzero remainder that is *not* a multiple of 10; eventually, there exists a value of k such that

$b_{2r-k} \neq a_k$. Thus, if $n \times n$ is improper, then $n \notin V$. This completes the proof of Theorem 1.

The theorem just proved greatly facilitates the search for validromic square roots (and validromic squares). A by-product of its proof is that if $n \in V$ and n has r digits, then n^2 has $2r - 1$ digits; to avoid trivial variants, we adopt the convention that, if $n = \theta_1\theta_2\ldots\theta_r \in V$, then $\theta_1 \neq 0$, $\theta_r \neq 0$. Thus, $n^2 < 10^{2r-1}$, which implies the following

Corollary: If $n \in V$ has r digits, then $n \le [10^{r-\frac{1}{2}}] = \underbrace{3162\ldots}_{(r \text{ digits})}$.

Let n_r denote the largest r-digit validromic square root. Then, by the Corollary, $n_1 \le 3$, $n_2 \le 31$, $n_3 \le 316$, etc. We readily find that $n_1 = 3$ (trivially), and $n_2 = 31$. There are other useful observations that may be made to facilitate extension of these initial results.

In what follows, we suppose that $n_r = \theta_1\theta_2\ldots\theta_r \in V$ (with the conventions as described previously). First, we surmise that $\theta_1 = 3$ for all r; this is easily proved. Clearly, this is true for $r = 1$ and $r = 2$. If $r > 2$, define

$$m_r = 3\,\underbrace{0\,0\,\ldots\,0}_{r-2}\,1;$$

then

$$m_r^2 = 9\,\underbrace{0\,0\,\ldots\,0}_{r-2}\,6\,\underbrace{0\,0\,\ldots\,0}_{r-2}\,1, \text{ and } (m_r')^2 = 1\,\underbrace{0\,0\,\ldots\,0}_{r-2}\,6\,\underbrace{0\,0\,\ldots\,0}_{r-2}\,9,$$

so $m_r \in V$. Since $n_r \ge m_r$, by definition of n_r, thus $\theta_1 \ge 3$. But the Corollary implies $\theta_1 \le 3$. Hence, $\theta_1 = 3$.

Clearly, if $n > m$ and $m \times m$ is improper, so is $n \times n$. This observation allows us to reject all candidates for n_r which exceed a previously excluded candidate and differ from it in only one or more digits. However, a much more powerful result may be inferred, which greatly reduces our search for n_r. Given that $\theta_1 = 3$ and $\theta_r = 1$, then the formula in (3) implies:

$$s_k \ge 2\theta_1\theta_k = 6\theta_k, \text{ for } k = 2, 3, \ldots, r.$$

However, s_k must be a digit; this implies $\theta_k = 0$ or 1. Therefore, n_r must be composed of "binary" digits, except for its leading digit, which is 3, and its last digit must equal 1.

Proceeding largely by trial and error, with the tools developed thus far, we find n_r, at least for the initial values of r. We begin from the left with $\theta_1 = 3$, then affix as many consecutive 1's as possible to the right. When one or more 0's must be used, we try to minimize the number of such 0's, and to push them as far to the right as possible, subject only to the condition that $\theta_r = 1$. As we proceed, we keep track of the rejected candidates, so as to reduce our search. Thus, if $\theta_1'\theta_2'\ldots\theta_r'$ is such a rejected value for n_r, then we know that $n_{r+1} < \theta_1'\theta_2'\ldots\theta_r'1$. Proceeding in this fashion, we find the following values of n_r, up to $r = 15$ (though we could have continued the table, by these same methods):

(5)

r	n_r	r	n_r	r	n_r
1	3	6	311101	11	31111 01000 1
2	31	7	31111 01	12	31111 01010 01
3	311	8	31111 001	13	31111 01010 001
4	3111	9	31111 0101	14	31111 01010 0011
5	31111	10	31111 01001	15	31111 01010 00001

Inspection of the foregoing table leads to the conjecture that θ_k is constant for all sufficiently large r; a rigorous proof of this premise seems possible but was not attempted. A related observation is that, for sufficiently large k, the values of θ_k do not affect the leading digits of n_r^2.

To stress dependence of r (as well as k), we use the expanded notation:

$$\theta_k^{(r)} \equiv \theta_k, \quad s_k^{(r)} \equiv s_k.$$

If r_k represents the minimum value of r such that $\theta_k^{(r)} = \theta_k$, a constant for all $r \geq r_k$, we can tabulate our *apparent* results as follows:

	k	r_k	θ_k	k	r_k	θ_k	k	r_k	θ_k	
	1	1	3	7	9	1	13	15	0	
	2	2	1	8	9	0	14	21	1	
(6)	3	3	1	9	12	1	15	16	0	
	4	4	1	10	11	0	16	17	0	
	5	7	1	11	12	0	17	26	1	
	6	7	0	12	16	1	18	19	0	etc.

Of course, in order to form this table, we first need to compute n_r for $r \gg 18$; even then, we cannot always be certain that the values in (6) are correct, at least for the higher values of k. However, if we *can* accept these values as gospel, we may then extend the table of n_r's.

The number of terms $\theta_i\theta_j$ in $s_k^{(r)}$ is maximized when $k = r$, and such number is r. A necessary (but not sufficient) test, therefore, is that $s_r^{(r)}$ be a digit. Other values of $s_r^{(r)}$ also need to be tested, of course; since the ones most likely to fail are the ones whose terms contain $\theta_1 = 3$, we test those first.

We illustrate by finding n_{27}, assuming that (6) is correct. We note that

$$s_{27}^{(27)} = \sum_{i=1}^{27} \theta_i^{(27)} \theta_{28-i}^{(27)} = 2\sum_{i=1}^{13} \theta_i \theta_{28-i}^{(27)} + \theta_{14}^2, \text{ with } \theta_{27}^{(27)} = 1;$$

thus,

$$s_{27}^{(27)} = 2(\theta_{27}^{(27)}\theta_1 + \theta_{26}^{(27)}\theta_2 + \theta_{25}^{(27)}\theta_3 + \theta_{24}^{(27)}\theta_4 + \theta_{23}^{(27)}\theta_5 + \theta_{21}^{(27)}\theta_7 + \theta_{19}^{(27)}\theta_9) + \theta_{14}^2$$

$$= 2(3 + \theta_{26}^{(27)} + \theta_{25}^{(27)} + \theta_{24}^{(27)} + \theta_{23}^{(27)} + \theta_{21}^{(27)} + \theta_{19}^{(27)}) + 1.$$

To maximize n_{27}, we may *attempt* $\theta_{19}^{(27)} = 1$; however, since $s_{27}^{(27)}$ is to be a digit, this forces $\theta_{21}^{(27)} = \theta_{23}^{(27)} = \theta_{24}^{(27)} = \theta_{25}^{(27)} = \theta_{26}^{(27)} = 0$. At this point, nothing can be inferred about $\theta_{20}^{(27)}$ or $\theta_{22}^{(27)}$; for this, we need to consider the following:

$$s_{20}^{(27)} = \sum_{i=1}^{20} \theta_i^{(27)}\theta_{21-i}^{(27)} = 2(\theta_{20}^{(27)}\theta_1 + \theta_{19}^{(27)}\theta_2 + \theta_{17}\theta_4 + \theta_{14}\theta_7 + \theta_{12}\theta_9)$$

$$= 2(\theta_{20}^{(27)} + 1 + 1 + 1 + 1),$$

assuming $\theta_{19}^{(27)} = 1$. In order for this last expression to be a digit, we must have $\theta_{20}^{(27)} = 0$. Likewise, we find that $\theta_{19}^{(27)} = 1$ implies $s_{22}^{(27)} = 2(\theta_{22}^{(27)} + 1 + 1 + 1 + 1)$, which can only be a digit if $\theta_{22}^{(27)} = 0$. Therefore, we surmise that n_{27} is given by using the values of θ_k shown by (6) for its first 18 digits, then, with $\theta_{19}^{(27)}\theta_{20}^{(27)}\ldots\theta_{27}^{(27)} = 100000001$. Testing this as a candidate for n_{27}, we find that it works; hence, we conclude that n_{27} is as just described.

Continuing in this fashion, we may extend (5) and (6) by alternating back and forth between tables. With considerable effort, the following additional values of n_r were derived (manually) by these methods:

r	n_r
16	31111 01010 01000 1
17	31111 01010 01010 01
18	31111 01010 01000 001
19	31111 01010 01010 0101
20	31111 01010 01000 00001
21	31111 01010 01010 01000 1
22	31111 01010 01010 01000 01
23	31111 01010 01010 01000 001
24	31111 01010 01010 01010 0001
25	31111 01010 01010 00000 00001
26	31111 01010 01010 01010 00010 1
27	31111 01010 01010 01010 00000 01
28	31111 01010 01010 01010 00000 001
29	31111 01010 01010 01010 00010 0001
30	31111 01010 01010 01000 10000 00001

In theory, one could extend these results indefinitely, however, without the aid of a computer, human endurance wanes. It seems quite plausible that a program might be devised, enabling extension of the foregoing tables to an arbitrary degree. The aim of such extension would be to discover any lurking pattern in the sequence of "binary" digits among the θ_k's, as k increases. It may be surmised that, having discovered such a pattern, one might be able to prove its validity rigorously. This exercise is left for the interested reader.

As for this particular solver, he gave up the effort at $r = 30$. Therefore, to "answer" both parts of the problem simultaneously (since neither n_r nor n_r^2, clearly, are palindromes), the largest validromic square found was n_{30}^2, where

$$n_{30} = 31111\ 01010\ 01010\ 01000\ 10000\ 00001.$$

Note: The proposer noted that $441 = 21 \cdot 21$, so that the restriction of factors to squares is unnecessary.

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